# Mean-field and beyond in the 2D BCS-BEC crossover

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# Summary

- Condensation and superfluidity in 2D systems
- 2D Fermi gas with pairing
- Mean-field
- Zero-temperature
- Finite-temperature
- Beyond mean-field
- Open problems

# Condensation and superfluidity in 2D systems

According to the **Mermin-Wagner theorem**[1] in a 2D uniform system one can find true condensation, i.e off-diagonal-long-range-order (ODLRO), only at zero temperature ($T = 0$).

Nevertheless, as shown by Hohenberg[2] the 2D uniform system can have quasi condensation, i.e. algebric-long-range-order (ALRO), below a critical finite temperature. This critical temperature is usually identified with the Berezinskii-Kosterlitz-Thouless temperature[3] below which the 2D system has a finite superfluidity.

[1] N.D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 133 (1966).
[2] P.C. Hohenberg, Phys. Rev. **158**, 383 (1967).
[3] V.L. Berezinskii, Sov. Phys. JEPT **34**, 610 (1972); J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973).

# 2D Fermi gas with pairing (I)

We consider a **2D neutral Fermi gas with attractive s-wave interaction**. The partition function $\mathcal{Z}$ of the system at temperature $T$, in a region of area $L^2$, and with chemical potential $\mu$ can be written as

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \ \exp\left\{ -\frac{1}{\hbar} \, S \right\} , \tag{1}$$

where

$$S = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r} \ \mathcal{L} \tag{2}$$

is the Euclidean action functional and $\mathcal{L}$ is given by

$$\mathcal{L} = (\bar{\psi}_\uparrow \ , \ \bar{\psi}_\downarrow) \left[ \hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} + g \, \bar{\psi}_\uparrow \, \bar{\psi}_\downarrow \, \psi_\downarrow \, \psi_\uparrow \tag{3}$$

with $g < 0$ is the attractive strength of the s-wave coupling. Notice that $\beta = 1/(k_B T)$ with $k_B$ the Boltzmann constant.

# 2D Fermi gas with pairing (II)

The Lagrangian density $\mathcal{L}$ is quartic in the fermionic fields $\psi_s$, but one can reduce the problem to a quadratic Lagrangian density by introducing an auxiliary complex scalar field $\Delta(\mathbf{r},\tau)$ via Hubbard-Stratonovich transformation[4], which gives

$$\mathcal{Z} = \int \mathcal{D}[\psi_s,\bar{\psi}_s]\,\mathcal{D}[\Delta,\bar{\Delta}]\ \exp\left\{-S_e/\hbar\right\}\ , \tag{4}$$

where

$$S_e = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r}\ \mathcal{L}_e \tag{5}$$

and the (exact) effective Euclidean Lagrangian density $\mathcal{L}_e$ reads

$$\mathcal{L}_e = (\bar{\psi}_\uparrow\ ,\ \bar{\psi}_\downarrow)\left[\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu\right]\left(\begin{array}{c}\psi_\uparrow\\ \psi_\downarrow\end{array}\right) + \bar{\Delta}\,\psi_\downarrow\,\psi_\uparrow + \Delta\bar{\psi}_\uparrow\,\bar{\psi}_\downarrow - \frac{|\Delta|^2}{g}\ . \tag{6}$$

[4] H.T.C. Stoof, K.B. Gubbels, D.B.M. Dickerscheid, Ultracold Quantum Fields (Springer, Dordrecht, 2009).

# 2D Fermi gas with pairing (III)

It is a standard procedure to integrate out the quadratic fermionic fields and to get a new effective action $S_{eff}$ which depends only on the auxiliary field $\Delta(\mathbf{r},\tau)$. In this way we obtain

$$\mathcal{Z} = \int \mathcal{D}[\Delta,\bar{\Delta}] \quad \exp\left\{-S_{eff}/\hbar\right\} , \tag{7}$$

where

$$S_{eff} = -Tr[\ln\left(G^{-1}\right)] - \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r}\, \frac{|\Delta|^2}{g} \tag{8}$$

with

$$G^{-1} = \begin{pmatrix} \hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu & \Delta \\ \bar{\Delta} & \hbar\partial_\tau + \frac{\hbar^2}{2m}\nabla^2 + \mu \end{pmatrix} \tag{9}$$

We stress that at this level the effective action $S_{eff}$ is formally exact.

# Mean-field (I)

In the mean-field approximation one consider a constant and real gap parameter, i.e.

$$\Delta(\mathbf{r},\tau) = \Delta_0 \ , \tag{10}$$

and the partition function becomes

$$\mathcal{Z}_{mf} = \exp\left\{-S_{mf}/\hbar\right\} = \exp\left\{-\beta\Omega_{mf}\right\} \ , \tag{11}$$

where

$$\Omega_{mf} = -\sum_{\mathbf{k}} \frac{1}{\beta}\left[2\ln(2\cosh(\beta E_k/2)) - \beta\xi_k\right] - L^2\frac{\Delta_0^2}{g} \tag{12}$$

with $\xi_k = \hbar^2 k^2/(2m) - \mu$ and

$$E_{\mathbf{k}} = \sqrt{\xi_k^2 + \Delta_0^2} \ . \tag{13}$$

# Mean-field (II)

The constant and real gap parameter $\Delta_0$ is obtained from

$$\frac{\partial \Omega_{mf}}{\partial \Delta_0} = 0 \ , \tag{14}$$

which gives the **gap equation**

$$-\frac{1}{g} = \frac{1}{L^2} \sum_{\mathbf{k}} \frac{\tanh\left(\beta E_{\mathbf{k}}/2\right)}{2E_{\mathbf{k}}} \ . \tag{15}$$

The integral on the right side of this equation is divergent. However, in two dimensions quite generally a bound-state energy $\epsilon_B$ exists. For the contact potential the bound-state equation is

$$-\frac{1}{g} = \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{1}{2\frac{\hbar^2 k^2}{2m} + \epsilon_B} \ . \tag{16}$$

# Mean-field (III)

In this way one obtains the **regularized gap equation**[5]

$$\sum_{\mathbf{k}} \left( \frac{\tanh\left(\beta E_{\mathbf{k}}/2\right)}{\frac{\hbar^2 k^2}{2m} + \frac{\epsilon_B}{2}} - \frac{1}{E_k} \right) = 0 \,, \tag{17}$$

which can be used to study the BCS-BEC crossover by varying the binding energy $\epsilon_B$.

We observe that the binding energy $\epsilon_B$ can be written as $\epsilon_B \simeq \hbar^2/(m a_{2D})$, where $a_{2D}$ is the 2D s-wave scattering length, such that $a_{2D} \simeq a_z \exp(-a_z/a_{3D})$ with $a_{3D}$ the 3D scattering length and $a_z$ the characteristic length of the transverse confinement.[6]

[5] M. Randeria, J-M. Duan, L-Y. Shieh, Phys. Rev. B **41**, 327 (1990).
[6] G. Bertaina and S. Giorgini, Phys. Rev. Lett. **106**, 110403 (2011).

# Mean-field (IV)

From the thermodynamic formula

$$N = -\left(\frac{\partial \Omega_{mf}}{\partial \mu}\right)_{L^2,T} \qquad (18)$$

we obtain the equation for the total number of fermions

$$N = \sum_{\mathbf{k}} \left(1 - \frac{\xi_k}{E_{\mathbf{k}}} \tanh\left(\beta E_{\mathbf{k}}/2\right)\right) . \qquad (19)$$

Moreover, the equation for the $T = 0$ number of quasi-condensed fermionic atoms[7] reads

$$N_0 = 2 \int d^2\mathbf{r}\, d^2\mathbf{r}' \, |\langle \psi_{\downarrow}(\mathbf{r}) \, \psi_{\uparrow}(\mathbf{r}') \rangle|^2 = \sum_{\mathbf{k}} \frac{\Delta_0^2}{2E_k^2} \tanh\left(\beta E_{\mathbf{k}}/2\right) . \qquad (20)$$

[7] LS, N. Manini, A. Parola, Phys. Rev. A **72**, 023621 (2005).

# Zero-temperature properties (I)

At $T = 0$ the grand potential is given by

$$\Omega_{mf} = -\frac{m}{4\pi\hbar^2} L^2 \left( \mu^2 + \mu \sqrt{\mu^2 + \Delta_0^2} \right) , \tag{21}$$

where the chemical potential $\mu$ reads

$$\mu = \epsilon_F - \frac{1}{2}\epsilon_B , \tag{22}$$

with $\epsilon_F = \pi\hbar^2 n/m$ the 2D Fermi energy, and the gap parameter $\Delta_0$ is instead

$$\Delta_0 = \sqrt{2\epsilon_F \epsilon_B} . \tag{23}$$

In addition, we find[8] this nice formula for the condensate fraction

$$\frac{N_0}{N} = \frac{1}{2} \frac{\frac{\pi}{2} + \arctan\left(\frac{\mu}{\Delta}\right)}{\frac{\mu}{\Delta} + \sqrt{1 + \frac{\mu^2}{\Delta^2}}} . \tag{24}$$

[8] LS, Phys. Rev. A **76**, 015601 (2007).

# Zero-temperature properties (II)

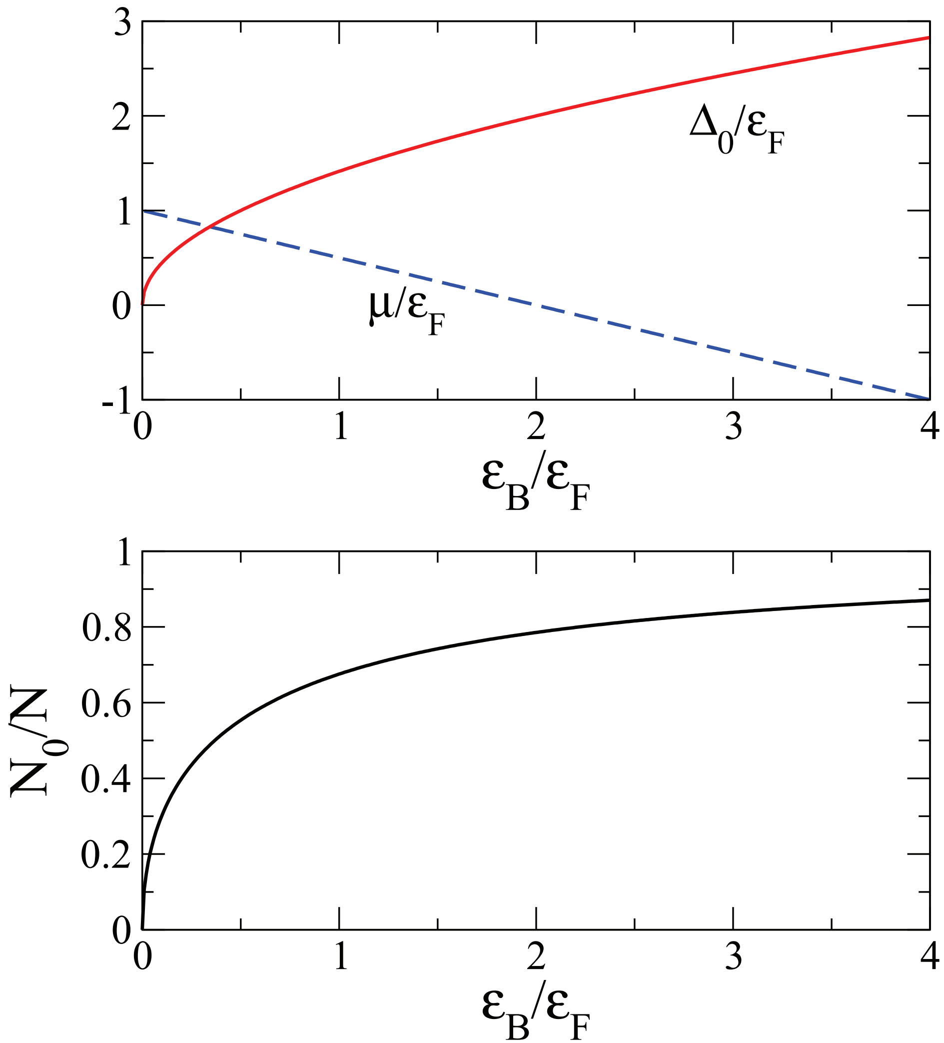


Figure: Upper panel: chemical potential $\mu$ and energy gap $\Delta_0$ as a function of the binding energy $\epsilon_B$ of pairs. Lower panel: Bose-condensate fraction $N_0/N$ of fermionic atoms as a function of the binding energy $\epsilon_B$ of pairs.

# Zero-temperature properties (III)

According to Landau[9] the first sound velocity $c_s$ is given by

$$m\, c_s^2 = \left( \frac{\partial P}{\partial n} \right)_{L^2, \bar{S}} \,, \tag{25}$$

where $P$ is the pressure and $\bar{S} = S/N$ is the entropy per particle of the superfluid. Moreover, at zero temperature it holds the following equality

$$\left( \frac{\partial P}{\partial n} \right)_{L^2, 0} = n \left( \frac{\partial \mu}{\partial n} \right)_{L^2} \,. \tag{26}$$

Using the 2D zero-temperature mean-field result

$$\mu = \epsilon_F - \frac{1}{2}\epsilon_B \,, \tag{27}$$

where $\epsilon_F = (\pi\hbar^2/m)n = m v_F^2/2$, we finally obtain

$$c_s = \frac{v_F}{\sqrt{2}} \,. \tag{28}$$

[9] L.D. Landau, Journal of Physics USSR **5**, 71 (1941).

# Finite-temperature properties (I)

One can explicitly calculate the temperature $T^*$ at which $\Delta_0 = 0$. In particular, one obtains[10] the following equations

$$\mu(T^*) = k_B T^* \, \ln\left(e^{\epsilon_F/(k_B T^*)} - 1\right), \tag{29}$$

$$\epsilon_B = k_B T^* \frac{\pi}{\gamma} \exp\left(-\int_0^{\mu(T^*)/(2k_B T^*)} \frac{\tanh(u)}{u}\, du\right), \tag{30}$$

which determine $T^*$ and $\mu(T^*)$ as a function of the binding energy $\epsilon_B$, with $\gamma = 1.781$.

[10] V.P. Gusynin, V.M. Loktev, and Sharapov, J. Exp. Theor. Phys. **88**, 685 (1999).

# Finite-temperature properties (II)

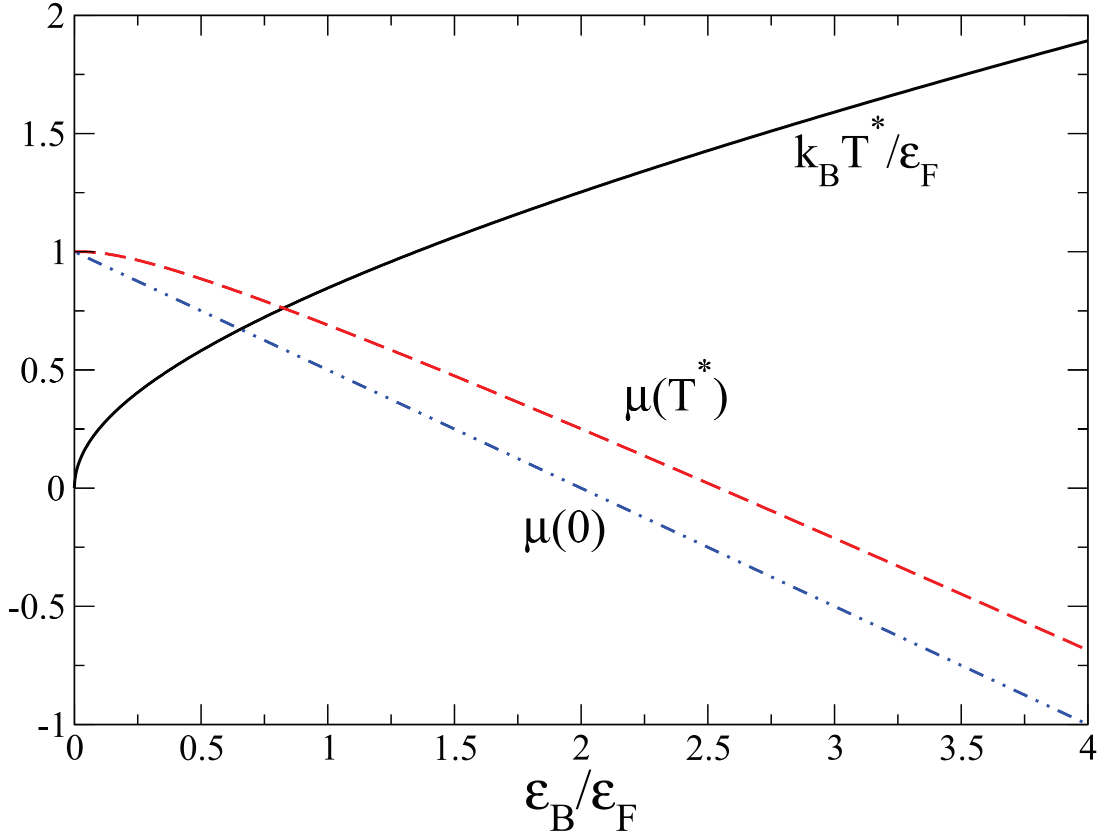


Figure: Critical temperature $T^*$ (solid line), critical chemical potential $\mu(T^*)$ (dashed line), and zero-temperature chemical potential $\mu(0)$ as a function of the binding energy $\epsilon_B$ of pairs.

# Beyond mean-field (I)

Let us now consider beyond mean-field effects. We have seen that the exact partition function can be written as

$$\mathcal{Z} = \int \mathcal{D}[\Delta, \bar{\Delta}] \quad \exp\left\{-S_{eff}[\Delta, \bar{\Delta}]/\hbar\right\} , \tag{31}$$

where $S_{eff}[\Delta, \bar{\Delta}]$ is the effective action, which is a functional of the complex bosonic auxiliary field $\Delta(\mathbf{r}, \tau)$ of pairing.
We impose that

$$\Delta(\mathbf{r}, \tau) = (\Delta_0 + \sigma(\mathbf{r}, \tau)) \; e^{i\theta(\mathbf{r},\tau)} . \tag{32}$$

The partition function can be then formally written as

$$\mathcal{Z} = e^{-\beta\Omega_{mf}(\Delta_0)} \int \mathcal{D}[\sigma, \theta] \quad \exp\left\{-S_{bmf}[\sigma, \theta; \Delta_0]/\hbar\right\} . \tag{33}$$

# Beyond mean-field (II)

Exanding $S_{bmf}[\sigma, \theta; \Delta_0]$ at the second order and functional-integrating over the amplitude field $\sigma(\mathbf{r}, \tau)$ one obtains[11]

$$\mathcal{Z} = e^{-\beta\Omega_{mf}(\Delta_0)} \int \mathcal{D}[\theta] \quad \exp\left\{-S_\theta[\theta; \Delta_0]/\hbar\right\} , \tag{34}$$

where

$$S_\theta[\theta; \Delta_0] = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r} \left\{ \frac{J}{2}(\nabla\theta)^2 + \frac{K}{2}(\partial_\tau \theta)^2 \right\} \tag{35}$$

is the action functional of the phase field (Goldstone field) with $J$ the phase stiffness and $K$ the phase susceptibility.
At $T = 0$ we find

$$J = \frac{\epsilon_F}{4\pi} , \qquad K = \frac{m}{4\pi} , \tag{36}$$

and the velocity $c_\theta$ of the Goldstone field reads

$$c_\theta = \sqrt{\frac{J}{K}} = \frac{v_F}{\sqrt{2}} = c_s . \tag{37}$$

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[11] A.M.J. Schakel, Ann. Phys. (N.Y.) **326**, 193 (2011).

# Beyond mean-field (III)

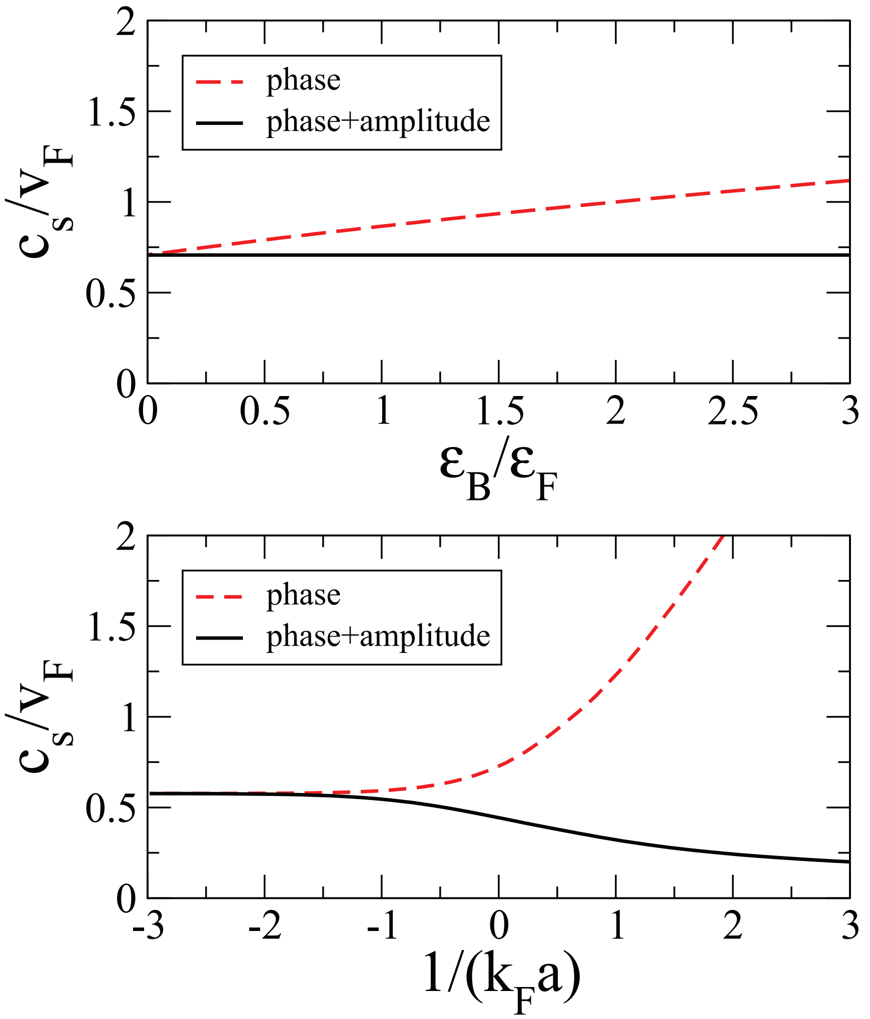



Figure: Upper panel: 2D scaled sound velocity $c_s/v_F$ vs scaled binding energy $\epsilon_B/\epsilon_F$. Lower panel: 3D scaled sound velocity $c_s/v_F$ vs scaled inverse interaction strength $1/(k_F a)$.

# Beyond mean-field (IV)

The renormalization-group theory[12] dictates that for our 2D system the superfluid density $n_s$ is zero above the **Berezinskii-Kosterlitz-Thouless critical temperature** $T_{BKT}$. Moveover below $T_{BKT}$ the superfluid density can be written as

$$n_s(T) = \frac{4m}{\hbar^2} J(T) \qquad \text{for } T < T_{BKT} , \tag{38}$$

and the critical temperature $T_{BKT}$ can be estimated by solving self-consistently

$$k_B T_{BKT} = \frac{\pi}{2} J(T_{BKT}) , \tag{39}$$

where $J(T)$ is the finite-temperature stiffness of our action functional $S_\theta$ of the phase.

[12] H.T.C. Stoof, K.B. Gubbels, D.B.M. Dickerscheid, Ultracold Quantum Fields (Springer, Dordrecht, 2009).

# Beyond mean-field (V)

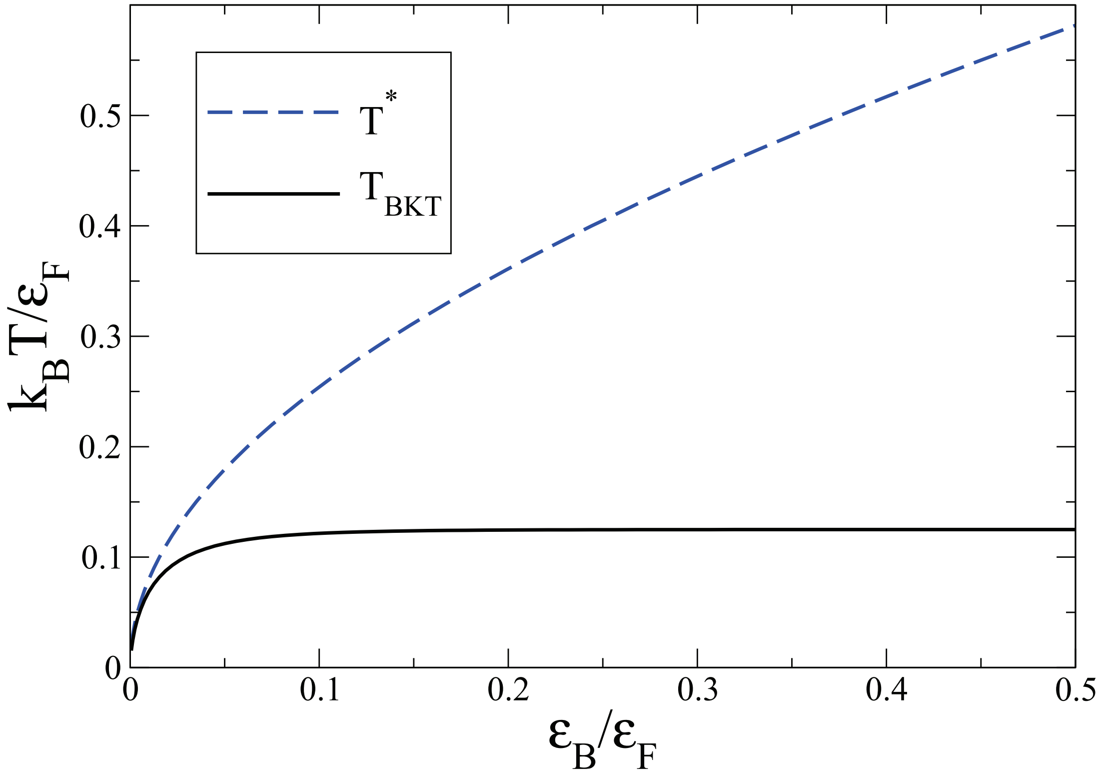



Figure: Dashed line: temperature $T^*$ above which $\Delta_0$ is zero; solid line: Berezinskii-Kosterlitz-Thouless critical temperature $T_{BKT}$.

# Beyond mean-field (VI)

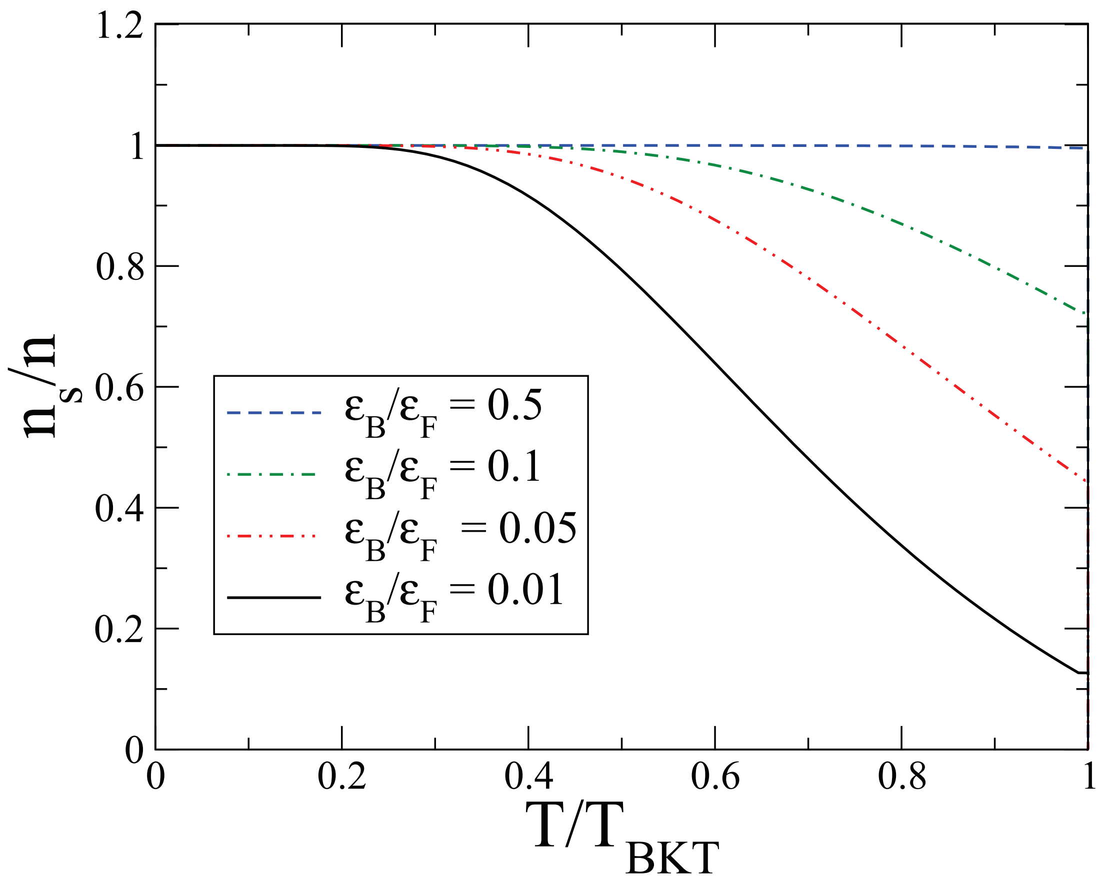



Figure: Superfluid fraction $n_s/n$ as a function of the scaled temperature $T/T_{BKT}$ for different values of the scaled binding energy $\epsilon_B/\epsilon_F$, where $\epsilon_F = (\hbar^2/m)\pi n$ is the Fermi energy. Above $T_{BKT}$ one has $n_s = 0$.

# Beyond mean-field (VII)

In our system the two-body density matrix

$$\rho_2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = \langle \bar{\psi}_\uparrow(\mathbf{r}_1,0)\,\bar{\psi}_\downarrow(\mathbf{r}_2,0)\,\psi_\downarrow(\mathbf{r}_3,0)\,\psi_\uparrow(\mathbf{r}_4,0)\rangle$$

shows **algebric-long-range-order** for $0 < T < T_{BKT}$.
In particular, introducing the center-of-mass positions of the two Cooper pairs, given by $\mathbf{R} = (\mathbf{r}_1+\mathbf{r}_2)/2$ and $\mathbf{R}' = (\mathbf{r}_3+\mathbf{r}_4)/2$, and their relative distances $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{r}' = \mathbf{r}_4 - \mathbf{r}_3$, for $|\mathbf{R}-\mathbf{R}'| \to \infty$ we find[1]

$$\rho_2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \simeq F^*(\mathbf{r})\,F(\mathbf{r}')\,\left(\frac{R_0}{|\mathbf{R}-\mathbf{R}'|}\right)^{\frac{k_B T}{8\pi J}}$$

where

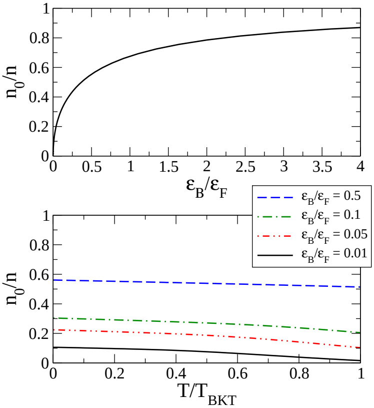
$$F(\mathbf{r}') = \frac{1}{L^2}\sum_{\mathbf{k}} \frac{\Delta_0}{2E_k}\tanh(\beta E_k/2)\,e^{i\mathbf{k}\cdot\mathbf{r}'}$$

is the mean-field wavefunction of the Cooper pair.

[1] LS, P.A. Marchetti, F. Toigo, arXiv:1309.7459.

The finite-temperature quasi-condensate density of atoms in the 2D superfluid is then given by

$$n_0 = 2 \int d^2\mathbf{r}' \, |F(\mathbf{r}')|^2 = \frac{\Delta_0^2}{2L^2} \sum_{\mathbf{k}} \frac{\tanh^2(\beta E_k/2)}{E_k^2}.$$



# Open problems

There are several open problems regarding our 2D Fermi superfluid in the BCS-BEC crossover. Among them we mention:

- first and second sound at finite temperature
- beyond mean-field equation of state: comparison with MC results
  Bertaina and Giorgini, PRL **106**, 110403 (2011)
- unbalanced system

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