Quantum effective action for Josephson dynamics

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Outline

- Bosonic Josephson tunneling
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Bosonic Josephson tunneling (I)

A system of N interacting bosons confined by a symmetric double-well potential can be described by the two-site Bose-Hubbard model

$$\hat{H} = -J\left(\hat{a}_1^+ \hat{a}_2 + \hat{a}_2^+ \hat{a}_1\right) + \frac{U}{2}[\hat{N}_1(\hat{N}_1 - 1) + \hat{N}_2(\hat{N}_2 - 1)] \qquad (1)$$

with $J > 0$ the tunneling (hopping) energy, U the boson-boson interaction, and $\hat{N}_j = \hat{a}_j^+ \hat{a}_j$. Here $\hat{a}_1$ and $\hat{a}_j^+$ are the bosonic ladder operators.

The mean-field approximation is obtained[1] by using Glauber coherent states

$$|\psi(t)\rangle = |\alpha_1(t)\rangle_1 \, |\alpha_2(t)\rangle_2 \qquad (2)$$

where $|\alpha_j(t)\rangle$ is the eigenstate of the annihilation operator $\hat{a}_j$, with complex eigenvalue

$$\alpha_j(t) = \sqrt{N_j(t)}\, e^{i\phi_j(t)} \;, \qquad (3)$$

where $N_j(t) = \langle\psi(t)|\hat{N}_j|\psi(t)\rangle$ is the average number of bosons in the site $j = 1, 2$ and $\phi_j(t)$ is the corresponding phase.

[1] R. Franzosi and V. Penna, Phys. Rev. E **67**, 046227 (2003).

Bosonic Josephson tunneling (II)

Quite remarkably, the mean-field dinamics is obtained by extremizing the following action functional

$$S = \int \langle\psi(t)| \left(i\hbar \frac{\partial}{\partial t} - \hat{H} \right) |\psi(t)\rangle. \quad (4)$$

One can also introduce[2] the relative phase

$$\phi(t) = \phi_2(t) - \phi_1(t) \quad (5)$$

and the normalized population imbalance

$$z(t) = \frac{N_1(t) - N_2(t)}{N} \qquad \in [-1, 1] \quad (6)$$

Here $N = N_1(t) + N_2(t)$ is a constant of motion.
In this framework $\phi(t)$ and $z(t)$ are the time-dependent variational parameters of the cherent state $|\psi(t) >$ which extremize the action S.

[2]A. Smerzi *et al.*, Phys. Rev. Lett. **79**, 4950 (1997).

Bosonic Josephson tunneling (III)

Specifically, we find[3]

$$S[z,\phi] = \int dt \left[\frac{N\hbar z}{2}\dot{\phi} - \frac{UN^2}{4} z^2 + JN\sqrt{1-z^2}\cos\phi \right] , \qquad (7)$$

with $\phi(t)$ and $z(t)$ Lagrangian variables. Actually, for this specific problem $z(t)$ and $\phi(t)$ are canonically conjugated.
The corresponding Euler-Lagrange equations are

$$\hbar\dot{\phi} = \frac{2Jz}{\sqrt{1-z^2}}\cos\phi + UNz + \varepsilon, \qquad (8a)$$

$$\hbar\dot{z} = -2J\sqrt{1-z^2}\sin\phi \, . \qquad (8b)$$

Linearizing around $z = 0$ and $\phi = 0$ one gets the Josephson frequency

$$\omega_J = \sqrt{2J(UN+2J)}/\hbar \, . \qquad (9)$$

This prediction was experimentally verified in 2005 with ^{87}Rb atoms.[4]

[3] S. Wimberger, G. Manganelli, A. Brollo, L.S., Phys. Rev. A **103**, 023326 (2021).
[4] M. Albiez *et al.*, Phys. Rev. Lett. **95**, 010402(2025).

Only-phase effective action

Given the action $S[z, \phi]$, the effective action for the phase $S[\phi]$ is defined as[5]

$$e^{\frac{i}{\hbar} S[\phi]} = \int \mathcal{D}[z] \, e^{\frac{i}{\hbar} S[z,\phi]} \, . \qquad (10)$$

The path integral can be computed explicitly expanding $S[z, \phi,]$ up to second order around $z = 0$. The resulting only-phase mean-field action is given by

$$S[\phi] = \int dt \left[\frac{m(\phi)}{2} \dot{\phi}^2 - V(\phi) \right] \, , \qquad (11)$$

where

$$m(\phi) = \frac{N\hbar^2}{2(UN + 2J\cos(\phi))} \, . \qquad (12)$$

$$V(\phi) = -JN\cos(\phi) \, . \qquad (13)$$

Quite remarkably, with the only-phase action $S[\phi]$ one recovers exactly the same mean-field Josephson frequency obtained with $S[z, \phi]$.

[5] K. Furutani, J. Tempere, L.S., Phys. Rev. B **105**, 134510 (2022).

Only-phase quantum effective action (I)

The one-loop quantum effective action[6]

$$\Gamma[\phi] = S[\phi] + \frac{i\hbar}{2} \mathrm{Tr} \ln \left(\frac{\delta^2 S}{\delta \eta^2}[\phi] \right) \tag{14}$$

provides a systematic way to include beyond-mean-field (quantum) fluctuations. At zero temperature we find[7]

$$\Gamma[\phi] = \int dt \left[\frac{m_{\rm eff}(\phi)}{2} \dot{\phi}^2 - V_{\rm eff}(\phi) \right] , \tag{15}$$

where

$$m_{\rm eff}(\phi) = m(\phi) + \frac{\hbar}{32} \frac{\left(\partial_\phi \Omega(\phi)^2\right)^2}{\Omega(\phi)^5} \tag{16}$$

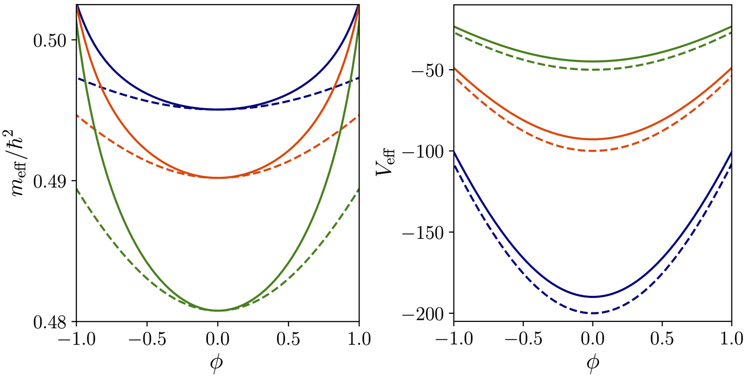
$$V_{\rm eff}(\phi) = V(\phi) + \frac{\hbar \Omega(\phi)}{2} \tag{17}$$

with

$$\Omega(\phi)^2 = \frac{V''(\phi) - \frac{m'(\phi)}{2m(\phi)} V'(\phi)}{m(\phi)} . \tag{18}$$

[6] S. Coleman, R. Jackiw, H.D. Politzer, Phys. Rev. D **10**, 2491 (1974).
[7] C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

Only-phase quantum effective action (II)



Effective mass (left panel) and effective potential as functions of ϕ for $U = J = 1.0$ and $N = 50$ (green lines), 100 (orange lines), and 200 (blue lines). The dashed lines represent the corresponding mean-field result. Adapted from C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

Only-phase quantum effective action (III)

Quantum corrections do not change the position of the minimum of the effective potential $V_{\rm eff}(\phi)$, which is still located at $\phi = 0$, where also $m'_{\rm eff}(0) = 0$. In particular, small oscillations around $\phi = 0$ are harmonic, with the frequency

$$\Omega_J = \sqrt{\frac{V''_{\rm eff}(0)}{m_{\rm eff}(0)}} = \omega_J \sqrt{1 - \frac{1}{2N} \frac{UN + 6J}{\sqrt{2J(UN+2J)}}}, \tag{19}$$

where

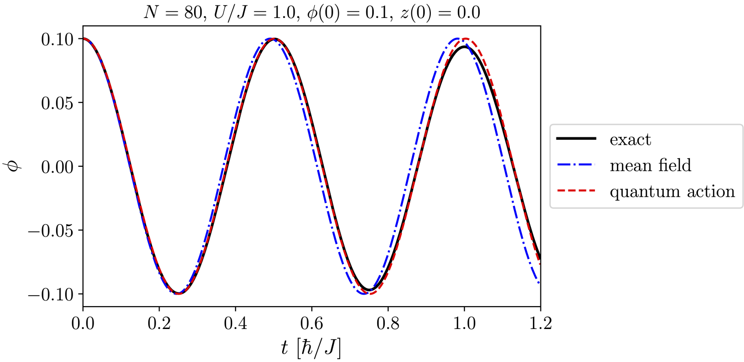
$$\omega_J = \frac{\sqrt{2J(UN+2J)}}{\hbar} \tag{20}$$

is the mean-field Josephson frequency.

- Exact numerical results[8] confirm the robustness of Eq. (19).
- The relative correction induced by quantum fuctuations can be of 3% for condensates with $N = 100$ atoms in realistic trapping configurations.

[8] C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

Only-phase quantum effective action (IV)



Comparison between the exact dynamics (solid black line), the mean-field dynamics (dashed-dotted blue line), and the quantum-corrected dynamics (dashed red line) of the relative phase, for $N = 80$, $U = J = 1.0$, $\phi(0) = 0.1$, and $\dot{\phi}(0) = 0$. Adapted from C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

Conclusions

- Quantum effective action: useful method for fields and dynamical variables.
- Provides a bridge between classical (or mean-field) dynamics and quantum fluctuations.
- Can include thermal effects perturbatively.
- Useful for theorists and experimentalists in quantum technologies.
- Work in progress: quantum effective action for optomechanics (with F. Lorenzi and M. Pelizzo).
- Work in progress: quantum effective action for resistively and capacitively shunted superconducting Josephson junction (with A. Bardin, K. Furutani, and J. Tempere).

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