# Bose-Einstein Condensation on Curved Surfaces

Luca Salasnich

Dipartimento di Fisica e Astronomia "Galileo Galilei", Università di Padova
Istituto Nazionale di Fisica Nucleare, Sezione di Padova
INO-CNR, Research Unit of Sesto Fiorentino, Consiglio Nazionale delle Ricerche

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# Summary

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- Non-interacting bosons: critical temperature
- Interacting bosons: thermodynamics
- Kosterlitz-Thouless transition
- Phase diagram for bosons on the surface of a sphere
- Conclusions and open problems

# Bose gas on the surface of a sphere (I)

Recently, Bose-Einstein condensates (BECs) made of ultracold alkali-metal atoms under microgravity have been achieved dropping the BEC down a 146-meter-long drop chamber[1] and rocketing the BEC and conducting experiments during in-space flight.[2]

In 2020 a BEC in harmonic trap[3] has been observed with the NASA's Cold Atom Laboratory onboard of International Space Station. Moreover, very recently the same group has reported the observation of ultracold atomic bubbles.[4]

[1] T. van Zoest, et al., Science **328**, 1540 (2010)
[2] D. Becker et al., Nature **562**, 391 (2018).
[3] D.C. Aveline et al., Nature **582**, 193 (2020).
[4] R.A. Carollo et al., e-preprint arXiv:2108.05880.

# Bose gas on the surface of a sphere (II)

Our theoretical study of a Bose gas on the surface of a sphere is triggered by the experimental confinement the atoms on a bubble trap,[5] which needs microgravity conditions.[6]
The energy of a particle of mass $m$ moving on the surface of a sphere of radius $R$ is quantized according to the formula

$$\epsilon_l = \frac{\hbar^2}{2mR^2} l(l+1) \, , \tag{1}$$

where $\hbar$ is the reduced Planck constant and $l = 0, 1, 2, ...$ is the **integer quantum number** of the angular momentum. This energy level has the degeneracy $2l + 1$ due to the magnetic quantum number $m_l = -l, -l+1, ..., l-1, l$ of the third component of the angular momentum.

[5] B. M. Garraway and H. Perrin, J. Phys. B **49**, 172001 (2016).
[6] E.R. Elliott et al., npj Microgravity **4**, 16 (2018); R.A. Carollo et al., e-preprint arXiv:2108.05880.

# Non-interacting bosons: critical temperature (I)

In quantum statistical mechanics the total number $N$ of non-interacting bosons moving on the surface of a sphere and at equilibrium with a thermal bath of absolute temperature $T$ is given by

$$N = \sum_{l=0}^{+\infty} \frac{2l+1}{e^{(\epsilon_l - \mu)/(k_B T)} - 1} \ , \qquad (2)$$

where $k_B$ is the Boltzmann constant and $\mu$ is the chemical potential. In the Bose-condensed phase, we can set[7] $\mu = 0$ and

$$N = N_0 + \sum_{l=1}^{+\infty} \frac{2l+1}{e^{\epsilon_l/(k_B T)} - 1} \ , \qquad (3)$$

where $N_0$ is the number of bosons in the lowest single-particle energy state, i.e. the number of bosons in the Bose-Einstein condensate (BEC).

[7] For details, see Martina Russo, BSc thesis, Supervisor: LS, Univ. of Padova (2019).

# Non-interacting bosons: critical temperature (II)

Within the semiclassical approximation, where $\sum_{l=1}^{+\infty} \to \int_1^{+\infty} dl$, the previous equation becomes

$$n = n_0 + \frac{mk_B T}{2\pi\hbar^2}\left(\frac{\hbar^2}{mR^2 k_B T} - \ln\left(e^{\hbar^2/(mR^2 k_B T)} - 1\right)\right), \qquad (4)$$

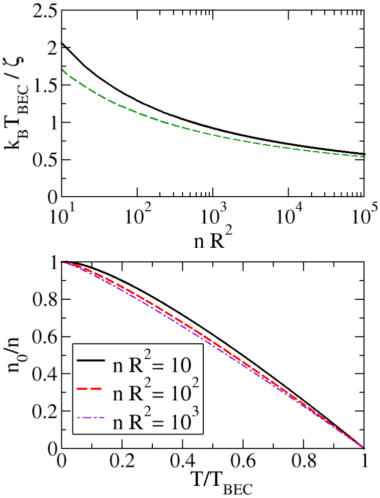
where $n = N/(4\pi R^2)$ is the 2D number density and $n_0 = N_0/(4\pi R^2)$ is the 2D condensate density.
At the critical temperature $T_{BEC}$, where $n_0 = 0$, one then finds[8]

$$k_B T_{BEC} = \frac{\frac{2\pi\hbar^2}{m} n}{\frac{\hbar^2}{mR^2 k_B T_{BEC}} - \ln\left(e^{\hbar^2/(mR^2 k_B T_{BEC})} - 1\right)}. \qquad (5)$$

As expected, in the limit $R \to +\infty$ one gets $T_{BEC} \to 0$, in agreement with the Mermin-Wagner theorem.[9] However, for any finite value of $R$ the critical temperature $T_{BEC}$ is larger than zero.

[8] A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).
[9] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

# Non-interacting bosons: critical temperature (III)




**Top panel**: $T_{BEC}$ vs $nR^2$, with $\zeta = \hbar^2 n/m$. Solid line: semiclassical approximation (solid line); dashed line: numerical evaluation of the sum. **Bottom panel**: condensate fraction $n_0/n$ vs temperature $T/T_{BEC}$.

# Interacting bosons: thermodynamics (I)

We now consider a system of interacting bosons on the surface of a sphere of radius R and contact interaction of strength $g$.
Within the formalism of functional integration, the grand canonical partition function reads

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}, \psi] \, e^{-\frac{S[\bar{\psi},\psi]}{\hbar}}, \quad (6)$$

where, by using $\beta = 1/(k_B T)$ with $T$ the absolute temperature,

$$S[\bar{\psi}, \psi] = \int_0^{\beta\hbar} d\tau \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\theta) \, d\theta \, R^2 \, \mathcal{L}(\bar{\psi}, \psi) \quad (7)$$

is the Euclidean action and, with $\hat{L}$ is the angular momentum operator,

$$\mathcal{L} = \bar{\psi}(\theta, \varphi, \tau) \left( \hbar\partial_\tau + \frac{\hat{L}^2}{2mR^2} - \mu \right) \psi(\theta, \varphi, \tau) + \frac{g}{2} |\psi(\theta, \varphi, \tau)|^4 \quad (8)$$

is the Euclidean Lagrangian of the bosonic field $\psi(\theta, \phi, \tau)$, which depends on the spherical angles $\theta$ and $\phi$ and on the imaginary time $\tau$.

# Interacting bosons: thermodynamics (II)

The condensate phase is introduced with the Bogoliubov shift

$$\psi(\theta,\varphi,\tau) = \psi_0 + \eta(\theta,\varphi,\tau), \tag{9}$$

where the real field configuration $\psi_0$ describes the condensate component. By substituting this field parametrization and keeping only second order terms in the field $\eta$ we rewrite the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g \tag{10}$$

with $\mathcal{L}_0 = -\mu{\psi_0}^2 + g{\psi_0}^4/2$.
We use the following decomposition of the complex fluctuation field $\eta(\theta,\varphi,\tau)$

$$\eta(\theta,\varphi,\tau) = \sum_{\omega_n}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l}\frac{e^{-i\omega_n\tau}}{R}\mathcal{Y}^l_{m_l}(\theta,\varphi)\,\eta(l,m_l,\omega_n), \tag{11}$$

where $\omega_n = 2\pi n/(\hbar\beta)$ are the Matsubara frequencies, and we introduce the orthonormal basis of the spherical harmonics $\mathcal{Y}^l_{m_l}(\theta,\phi)$.

# Interacting bosons: thermodynamics (III)

After some analytical calculations, at the Gaussian level the grand potential

$$\Omega = -\frac{1}{\beta}\ln(\mathcal{Z}) \simeq -\frac{1}{\beta}\left(\ln(\mathcal{Z}_0) + \ln(\mathcal{Z}_g)\right) \tag{12}$$

is given by

$$\begin{aligned}\Omega(\mu, {\psi_0}^2) &= 4\pi R^2\big(-\mu{\psi_0}^2 + g{\psi_0}^4/2\big) + \frac{1}{2}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} E_l(\mu, {\psi_0}^2) \\ &+ \frac{1}{\beta}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} \ln(1 - e^{-\beta E_l(\mu, {\psi_0}^2)})\end{aligned} \tag{13}$$

where

$$E_l(\mu, {\psi_0}^2) = \sqrt{(\epsilon_l - \mu + 2g{\psi_0}^2)^2 - g^2{\psi_0}^4} \tag{14}$$

is the excitation spectrum of the interacting system, with $\epsilon_l = \hbar^2 l(l+1)/(2mR^2)$ the single-particle energy.

# Interacting bosons: thermodynamics (IV)

The condensate number density $n_0$ of the system is given by

$$n_0 = {\psi_0}^2 \ , \tag{15}$$

where we fix the value of the order parameter $\psi_0$ with the condition

$$\frac{\partial \Omega(\mu, {\psi_0}^2)}{\partial \psi_0} = 0 \ . \tag{16}$$

Notice that from this formula we get $n_0$ as a function of $\mu$.
The total number density of the system is instead given by

$$n = -\frac{1}{4\pi R^2} \frac{\partial \Omega(\mu, n_0(\mu))}{\partial \mu} \ . \tag{17}$$

At the lowest order of a perturbative scheme,[10] where $\psi_0$ is obtained from the mean-field equation $\frac{\partial \Omega_0(\mu, {\psi_0}^2)}{\partial \psi_0} = 0$, we get $\psi_0 \simeq \sqrt{\mu/g}$ and

$$E_l \simeq E_l^B = \sqrt{\epsilon_l(\epsilon_l + 2\mu)} \ . \tag{18}$$

[10] H. Kleinert, S. Schmidt, and A. Pelster, Phys. Rev. Lett. **93**, 160402 (2004).

# Interacting bosons: thermodynamics (V)

Within this perturbative scheme[11] from the previous equations we obtain[12] the BEC critical temperature

$$k_B T_{BEC} = \frac{\frac{2\pi\hbar^2 n}{m} - \frac{gn}{2}}{\frac{\hbar^2}{2mR^2 k_B T_{BEC}}\left(1+\sqrt{1+\frac{2gmnR^2}{\hbar^2}}\right) - \ln\left(e^{\frac{\hbar^2}{mR^2 k_B T_{BEC}}\sqrt{1+\frac{2gmnR^2}{\hbar^2}}} - 1\right)}, \quad (19)$$

where the condensate density $n_0$ is zero.
Moreover, adopting the Landau formula for the normal density, we calculate the superfluid density $n_s(T)$ as

$$n_s = n - \frac{1}{k_B T}\int_1^{+\infty} \frac{dl\,(2l+1)}{4\pi R^2}\,\frac{\hbar^2(l^2+l)}{2mR^2}\,\frac{e^{E_l^B/(k_B T)}}{(e^{E_l^B/(k_B T)}-1)^2} \,. \quad (20)$$

and applying the Kosterlitz-Nelson criterion we evaluate numerically the BKT critical temperature $T_{BKT}$.

[11] H. Kleinert, S. Schmidt, and A. Pelster, Phys. Rev. Lett. **93**, 160402 (2004).
[12] A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).

# Kosterlitz-Thouless transition (I)

BEC phase transition: For an ideal gas of non-interacting identical bosons in a infinite volume (thermodynamic limit) there is BEC only below a critical temperature $T_{BEC}$. In particular one finds

$$k_B\, T_{BEC} = \begin{cases} \frac{2\pi}{\zeta(3/2)^{2/3}} \frac{\hbar^2}{m} n^{2/3} & \text{for } D = 3 \\ 0 & \text{for } D = 2 \\ \text{no solution} & \text{for } D = 1 \end{cases}$$

where $D$ is the spatial dimension of the system, $n$ is the number density, and $\zeta(x)$ is the Riemann zeta function.

This result due to Einstein (1925), which says that there is no BEC at finite temperature for $D \leq 2$ in the case of non-interacting bosons, was extended to interacting systems by **David Mermin** and **Herbert Wagner** in 1966.

# Kosterlitz-Thouless transition (II)

Despite the absence of BEC in 2D, in 1972 **Kosterlitz** and **Thouless** (but also **Vadim Berezinskii** (1935-1980)) suggested that a 2D fluid can be superfluid below a critical temperature, the so-called Berezinskii-Kosterlitz-Thouless critical temperature $T_{BKT}$.
The crucial issue is that the complex bosonic field can be written as

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})| \; e^{i\theta(\mathbf{r})} \tag{21}$$

where the angular field

$$\theta(\mathbf{r}) = \theta_0(\mathbf{r}) + \theta_v(\mathbf{r}) \; ,$$

is the sum of two controbutions: $\theta_0(\mathbf{r})$ has zero circulation (no vortices) while $\theta_v(\mathbf{r})$ encodes the contribution of quantized vortices.
In 2D the quantized vortices located at position $\mathbf{r}_i$ with quantum numbers $q_i$, interact through a 2D Coulomb-like potential

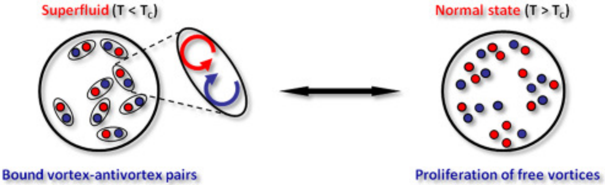
$$V(r) = -2\pi J \ln\left(\frac{r}{\xi}\right) \; ,$$

where $J = \frac{\hbar^2}{m} n_s$ is the phase stiffness, $\xi$ is healing length, i.e. the cutoff length defining the vortex core size, and $\mu_c$ the energy associated to the creation of a vortex.

# Kosterlitz-Thouless transition (III)

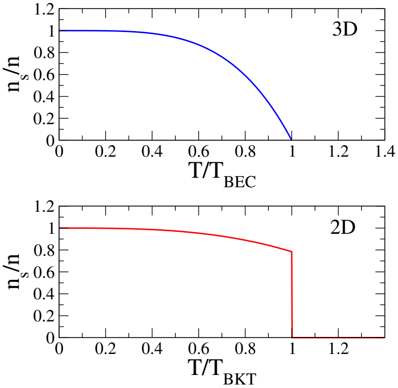
The analysis of **Kosterlitz** and **Thouless** on this 2D Coulomb-like gas of quantized vortices has shown that:

- As the temperature $T$ increases vortices start to appear in vortex-antivortex pairs (mainly with $q = \pm 1$).
- The pairs are bound at low temperature until at the critical temperature $T_c = T_{BKT}$ an unbinding transition occurs above which a proliferation of free vortices and antivortices is predicted.
- The phase stiffness $J$ and the vortex energy $\mu_c$ are renormalized.
- The renormalized superfluid density $n_s = J(m/\hbar^2)$ decreases by increasing the temperature $T$ and jumps to zero above $T_c = T_{BKT}$.



# Kosterlitz-Thouless transition (IV)

An important prediction of the Kosterlitz-Thouless transition is that, contrary to the 3D case, in 2D the superfluid fraction $n_s/n$ jumps to zero above a critical temperature.




For 3D superfluids the transition to the normal state is a BEC phase transition, while in 2D superfluids the transition to the normal state is something different: a topological phase transition.

# Phase diagram for bosons on the surface of a sphere (I)

In our problem of interacting bosons on the surface of a sphere, we determine the critical temperature $T_{BKT}$ by using the Nelson-Kosterlitz criterion[13]:

$$k_B T_{BKT} = \frac{\pi}{2} \frac{\hbar^2}{m} n_s(T_{BKT}^-) \ . \tag{22}$$

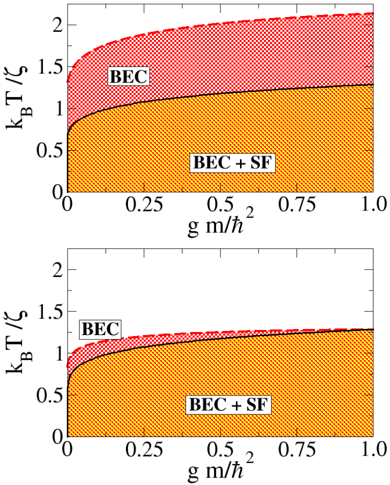
However, for the sake of simplicity we use the bare superfluid density $n_S(T)$ instead of the renormalized one.

In a very recent paper[14] we have instead used the renormalized superfluid density to determine $T_{BKT}$ by solving the Kosterlitz-Thouless renormalization group equations.

[13] D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. **39**, 1201 (1977).

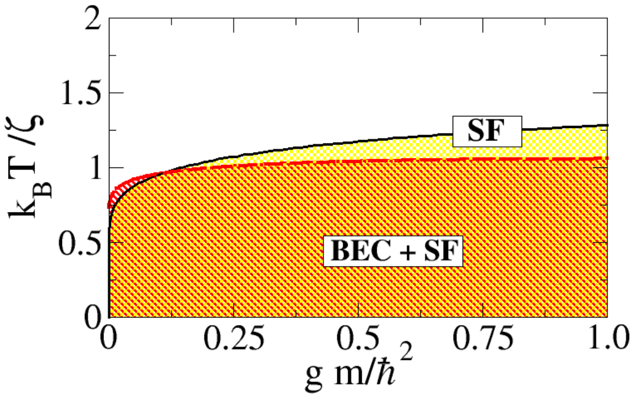
[14] A. Tononi, A. Pelster, and LS, e-preprint arXiv:2104.04585.

# Phase diagram for bosons on the surface of a sphere (II)




Phase diagram of the bosonic system for $nR^2 = 10^2$ (**upper panel**) and $nR^2 = 10^4$ (**lower panel**). Here $\zeta = \hbar^2 n/m$. Adapted from A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).

# Phase diagram for bosons on the surface of a sphere (III)




Phase diagram of the bosonic system for $nR^2 = 10^5$. Here $\zeta = \hbar^2 n/m$.
Adapted from A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).

# Conclusions and open problems (I)

- Triggered by recent achievements of space-based BECs under microgravity and bubble traps, which confine atoms on a thin shell, we have investigated[15] BEC on the surface of a sphere finding:
  – BEC critical temperature for non-interacting bosons;
  – BEC thermodynamcs, superfluid density, and BEC and BKT critical temperatures for interacting bosons.
- In a recent paper[16], we have instead analyzed BEC on the surface of an ellipsoid for realistic bubble-trap parameters calculating:
  – BEC critical temperature both non-interacting and interacting bosons;
  – the free expansion of the hollow Bose condensate.
- Finally, in a very recent paper[17] we have analyzed in detail the BKT phase transition for a BEC on the surface of a sphere calculating the renormalized superfluid density of the system by deriving and solving generalized Kosterlitz-Thouless renormalization group equations.

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[15] A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).
[16] A. Tononi, F. Cinti, and LS, Phys. Rev. Lett. **125**, 010402 (2020).
[17] A. Tononi, A. Pelster, and LS, e-preprint arXiv:2104.04585.

# Conclusions and open problems (II)

- The surface of a sphere has a constant curvature while the surface of an ellipsoid does not have a constant curvature. Does a locally-varying curvature affect the quantum-thermal properties of a Bose gas constrained to move on the surface of an ellipse?
- For a particle constrained on a curve it appears a quantum-curvature potential[18]

$$U_{QC}(s) = -\frac{\hbar^2 \kappa(s)^2}{8m} \, ,$$

  where $\kappa(s)$ is the local geodesic curvature of the curve and $s$ is the curvilinar abscissa (arclength) along the curve.
- Similarly, also for a particle constrained on a surface it appears a quantum-curvature potential.[19] In the case of the surface of an ellipsoid this quantum-curvature potential could strongly affect the quantum-thermal properties of a Bose gas.

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[18] P. Leboeuf and N. Pavloff, Phys. Rev. A **64**, 033602 (2001).

[19] N.S. Moller, F.E.A. dos Santos, V.S. Bagnato, and A. Pelster, New. J. Phys. **22**, 063059 (2020).

# Acknowledgements

**Thank you for your attention!**