Bright and dark solitons in nonlinear wave equations

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SERB School on Nonlinear Dynamics,
IIT Patna, December 2019

Summary

- Gross-Pitaevskii equation
- Dimensional reduction: from 3D to 1D
- 1D bright solitons
- 1D black solitons
- 1D gray solitons
- Conclusions

Gross-Pitaevskii equation (I)

Static and dynamical properties of a pure Bose-Einstein condensate made of dilute and ultracold atoms are very well described by the Gross-Pitaevskii equation[1]

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + (N-1)\frac{4\pi\hbar^2 a_s}{m}|\psi(\mathbf{r},t)|^2 \right] \psi(\mathbf{r},t) \; , \tag{1}$$

where $U(\mathbf{r})$ is the external trapping potential and a_s is the s-wave scattering length of the inter-atomic potential.
Here $\psi(\mathbf{r},t)$ is the wavefunction of the Bose-Einstein condensate normalized to one, i.e.

$$\int |\psi(\mathbf{r},t)|^2 \; d^3\mathbf{r} = 1 \; , \tag{2}$$

and such that $n(\mathbf{r}) = N|\psi(\mathbf{r},t)|^2$ is the local number density of the N condensed atoms.

[1] E.P. Gross, Nuovo Cimento **20**, 454 (1961); L.P. Pitaevskii, Sov. Phys. JETP. **13**, 451 (1961).

Dimensional reduction: from 3D to 1D (I)

The Gross-Pitaevskii (GP) equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + (N-1)g|\psi(\mathbf{r},t)|^2 \right] \psi(\mathbf{r},t) \ , \qquad (3)$$

with

$$g = \frac{4\pi\hbar^2}{m} a_s \ , \qquad (4)$$

is the Euler-Lagrange equation of the GP action functional

$$S = N \int dt\, d^3\mathbf{r}\, \psi^*(\mathbf{r},t) \Big(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - U(\mathbf{r}) - \frac{N-1}{2} g|\psi(\mathbf{r},t)|^2 \Big) \psi(\mathbf{r},t) \ . \qquad (5)$$

Let us now consider a very strong harmonic confinement of frequency $\omega_\perp$ along x and y and a generic confinement $\mathcal{U}(z)$ along z, namely

$$U(\mathbf{r}) = \frac{1}{2} m {\omega_\perp}^2 (x^2 + y^2) + \mathcal{U}(z) \ . \qquad (6)$$

Dimensional reduction: from 3D to 1D (II)

On the basis of the chosen external confinement, we adopt the ansatz

$$\psi(\mathbf{r}, t) = f(z, t) \frac{1}{\pi^{1/2} a_\perp} \exp\left(\frac{x^2 + y^2}{2 {a_\perp}^2} \right) , \quad (7)$$

where $f(z, t)$ is the axial wave function and $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$ is the characteristic length of the transverse harmonic confinement.
By inserting Eq. (7) into the GP action (5) and integrating along x and y, the resulting effective action functional depends only on the field $f(z, t)$.
One easily finds that the Euler-Lagrange equation of the axial wavefunction $f(z, t)$ reads

$$i\hbar \frac{\partial}{\partial t} f(z, t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \mathcal{U}(z) + \gamma |f(z, t)|^2 \right] f(z, t) , \quad (8)$$

where

$$\gamma = \frac{(N-1) g}{2\pi {a_\perp}^2} \quad (9)$$

is the effective one-dimensional interaction strength and the additive constant $\hbar\omega_\perp$ has been omitted because it does not affect the dynamics.

1D bright solitons (I)

In the absence of axial confinement, i.e. $\mathcal{U}(z) = 0$, the 1D GPE becomes

$$i\hbar \frac{\partial}{\partial t} f(z,t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \gamma |f(z,t)|^2 \right] f(z,t) \, . \tag{10}$$

This is a 1D nonlinear Schrödinger equation (NLSE) with cubic nonlinearity.
In 1972 Vladimir Zakharov and Aleksei Shabat[2] found that this equation admits solitonic solutions, such that

$$f(z,t) = \phi(z - vt) e^{i(mv^2/2 - \mu)t/\hbar} , \tag{11}$$

where v is the arbitrary velocity of propagation of the solution, which has a shape-invariant axial density profile:

$$n(z,t) = N|f(z,t)|^2 = N|\phi(z - vt)|^2 \, . \tag{12}$$

[2] V.E. Zakharov and A.B. Shabat, Sov. Phys. JETP **34**, 62 (1972).

1D bright solitons (II)

Setting $\zeta = z - vt$, the 1D stationary NLSE

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{d\zeta^2} + \gamma |\phi(\zeta)|^2 \right] \phi(\zeta) = \mu \, \phi(\zeta) \ , \tag{13}$$

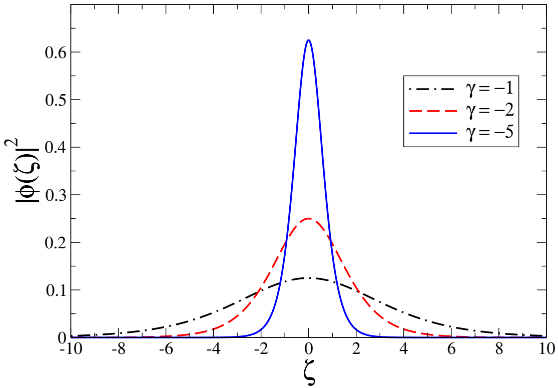
with $\gamma < 0$ (self-focusing), admits the bright soliton solution

$$\phi(\zeta) = \sqrt{\frac{m|\gamma|}{8\hbar^2}} \, Sech \left[\frac{m|\gamma|}{4\hbar^2} \zeta \right] \tag{14}$$

with $Sech[x] = \frac{2}{e^x + e^{-x}}$ and

$$\mu = -\frac{m \, \gamma^2}{16 \, \hbar^2} \ . \tag{15}$$

1D bright solitons (III)



Probability density $|\phi(\zeta)|^2$ of the bright soliton for three values of the nonlinear strength γ. We set $\hbar = m = 1$.

1D bright solitons (IV)

We now derive the analytical formula of the bright soliton.
Let us assume that $\phi(\zeta)$ is real. Then the 1D stationary NLSE can be rewritten as

$$\phi''(\zeta) = -\frac{\partial W(\phi)}{\partial \phi} \, , \tag{16}$$

where

$$W(\phi) = \frac{1}{2}\frac{m|\gamma|}{\hbar^2}\phi^4 + \frac{m\mu}{\hbar^2}\phi^2 \, . \tag{17}$$

Thus, $\phi(\zeta)$ can be seen as the "coordinate" for a fictitious particle at "time" ζ. The constant of motion of the problem reads

$$K = \frac{1}{2}\phi'(\zeta)^2 + W(\phi) \, , \tag{18}$$

from which one finds

$$\frac{d\phi}{d\zeta} = \sqrt{2(K - W(\phi))} \, . \tag{19}$$

1D bright solitons (V)

Imposing that $\phi(\zeta) \to 0$ as $|\zeta| \to \infty$ one gets $K = 0$ and consequently

$$\frac{d\phi}{\sqrt{-2W(\phi)}} = d\zeta \ , \tag{20}$$

or explicity

$$\frac{d\phi}{\sqrt{-\frac{m|\gamma|}{\hbar^2}\phi^4 + \frac{2m|\mu|}{\hbar^2}\phi^2}} = d\zeta \ , \tag{21}$$

with $\mu < 0$. Inserting the integrals one obtains

$$\int_{\phi(0)}^{\phi(\zeta)} \frac{d\phi}{\phi\sqrt{-\frac{m|\gamma|}{\hbar^2}\phi^2 + \frac{2m|\mu|}{\hbar^2}}} = \zeta \ . \tag{22}$$

Setting $\phi'(0) = 0$, from the definition of K and using $K = 0$ one finds $W(\phi(0)) = 0$ and therefore

$$\phi(0) = \sqrt{\frac{2|\mu|}{|\gamma|}} \ . \tag{23}$$

1D bright solitons (VI)

After integration of Eq. (22) one gets

$$\frac{1}{\sqrt{\frac{m|\mu|}{\hbar}}} ArcSech\left[\sqrt{\frac{|\gamma|}{2|\mu|}}\phi(\zeta)\right] = \zeta \tag{24}$$

from which

$$\phi(\zeta) = \sqrt{\frac{2|\mu|}{|\gamma|}} Sech\left[\sqrt{\frac{m|\mu|}{\hbar^2}}\zeta\right] \ . \tag{25}$$

Finally, imposing the normalization condition

$$\int d\zeta \ \phi(\zeta)^2 = 1 \ , \tag{26}$$

one obtains

$$\mu = -\frac{m \ \gamma^2}{16 \ \hbar^2} \ . \tag{27}$$

1D black solitons (I)

From the stationary 1D NLSE equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\gamma|\phi(z)|^2\right]\phi(z)=\mu\;\phi(z)\;. \tag{28}$$

one can also obtain other solitonic solutions.
Under the assumption $\gamma > 0$ (self-defocusing), we shall find that the 1D GPE admits the **black soliton** solution

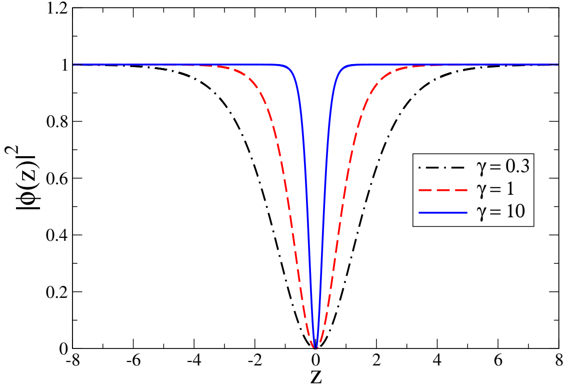
$$\phi(z)=\bar{\phi}\;Tanh\left[\sqrt{\frac{m\gamma}{\hbar^2}}\bar{\phi}\,z\right] \tag{29}$$

with $\bar{\phi}$ the value of the field $\phi(z)$ for $z\to+\infty$, $Tanh[x]=\frac{1-e^{-2x}}{1+e^{-2x}}$, and

$$\mu=\gamma\bar{\phi}^2\;. \tag{30}$$

Notice that one can prove that the **black soliton** is dynamically stable only if the velocity v of propagation is zero. That is way we consider here the case $v=0$.

1D black solitons (II)



Probability density $|\phi(z)|^2$ of the **black soliton** for three values of the nonlinear strength γ. We set $\hbar = m = \bar{\phi} = 1$.

1D black solitons (III)

We now derive the analytical formula of the **black soliton**.
Let us assume that $\phi(z)$ is real. Then the 1D stationary Gross-Pitaevskii equation can be rewritten as

$$\phi''(z) = -\frac{\partial W(\phi)}{\partial \phi} \ , \tag{31}$$

where

$$W(\phi) = -\frac{1}{2}\frac{m\gamma}{\hbar^2}\phi^4 + \frac{m\mu}{\hbar^2}\phi^2 \ . \tag{32}$$

Thus, as previously, $\phi(z)$ can be seen as the "coordinate" for a fictitious particle at "time" z. The constant of motion of the problem reads

$$K = \frac{1}{2}\phi'(z)^2 + W(\phi) \ , \tag{33}$$

from which one finds

$$\frac{d\phi}{dz} = \sqrt{2(K - W(\phi))} \ . \tag{34}$$

1D black solitons (IV)

Imposing that $\phi(z) \to \pm\bar{\phi}$ as $z \to \pm\infty$ one gets

$$K = W(\bar{\phi}) = -\frac{1}{2}\frac{m\gamma}{\hbar^2}\bar{\phi}^4 + \frac{m\mu}{\hbar^2}\bar{\phi}^2 = \frac{m\gamma}{2\hbar^2}\bar{\phi}^4 \tag{35}$$

because, from $\frac{\partial W}{\partial \phi}(\bar{\phi}) = 0$, one also finds

$$\mu = \gamma\bar{\phi}^2 \; . \tag{36}$$

In this way, the equation

$$\frac{d\phi}{dz} = \sqrt{2(K - W(\phi))} \tag{37}$$

can be rewritten as

$$\frac{d\phi}{\sqrt{\frac{m\gamma}{\hbar^2}(\bar{\phi}^2 - \phi^2)^2}} = dz \; . \tag{38}$$

Inserting the integrals one obtains

$$\int_0^{\phi(z)} \frac{d\phi}{\sqrt{\frac{m\gamma}{\hbar^2}}(\bar{\phi}^2 - \phi^2)} = z \tag{39}$$

setting $\phi(0) = 0$.

1D black solitons (V)

After integration of Eq. (39) one gets

$$\frac{1}{\sqrt{\frac{m\gamma}{\hbar^2}}} \frac{1}{\bar{\phi}} ArcTanh[\frac{\phi(z)}{\bar{\phi}}] = z \tag{40}$$

from which

$$\phi(z) = \bar{\phi} \; Tanh\left[\sqrt{\frac{m\gamma}{\hbar^2}} \bar{\phi} \, z \right] \; . \tag{41}$$

This is the **black soliton** solution. Remember that we have also found that

$$\mu = \gamma \bar{\phi}^2 \; . \tag{42}$$

1D gray solitons (I)

The density profile of the **black soliton** has a mimimum with zero density. There are, however, other dark solitonic solutions with a nonzero minimum.

They are called gray solitons and they are solutions of the time-dependent 1D NLSE

$$i\hbar\frac{\partial}{\partial t}f(z,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \gamma|f(z,t)|^2\right] f(z,t)\ . \tag{43}$$

In this case the wavefunction $f(z,t)$ is complex and it is given by

$$f(z,t) = \sqrt{\bar{f}^2 - f_0^2}\ Tanh[\sqrt{\frac{m\gamma}{\hbar^2}}\sqrt{\bar{f}^2 - f_0^2}(z - vt)] + i\, f_0\ , \tag{44}$$

where $\bar{f}$ is the value of the field $f(z,t)$ for $z \to +\infty$, f_0^2 is the minimal density, such that

$$f_0 = \frac{v}{v_s}\bar{f} \tag{45}$$

with $v_s = \bar{f}\sqrt{\gamma/m}$ the speed of sound.

Conclusions

- The 1D bright soliton analytical solution has been obtained from the 1D GPE, which is derived from the 3D GPE assuming a transverse Gaussian with a constant transverse width $a_\perp$.
- Also the **1D black soliton** analytical solution has been derived from the 1D GPE.
- Finally, we have seen that, in addition to the **1D black soliton**, there are other dark solutions, which are called 1D gray solitons.

Acknowledgements

Thank you for your attention!

Slides online: http://materia.dfa.unipd.it/salasnich/talk-patna19c.pdf