Photon Correlations at Thermal Equilibrum

Luca Salasnich

Dipartimento di Fisica "Galileo Galilei" and Padua QTech, Università di Padova
INFN, Sezione di Padova,
INO-CNR, Unit of Sesto Fiorentino

Padua, July 4, 2024
Meeting "Cosmology and Detection of Photons"

In collaboration with S. Matarrese, A. Ortolan and M. Toffoli.

Summary

- Gas of photons at thermal equilibrium
- Field operator for photons
- First order correlations
- Second order correlations
- Detecting time correlations
- Conclusions

Gas of photons at thermal equilibrium (I)

Let us consider the quantum radiation field in thermal equilibrium with a bath at the temperature T. The relevant quantity to calculate all thermodynamical properties of the system is the grand-canonical partition function $\mathcal{Z}$, given by

$$\mathcal{Z} = Tr[e^{-\beta(\hat{H} - \mu\hat{N})}] \tag{1}$$

where $\beta = 1/(k_B T)$ with $k_B = 1.38 \cdot 10^{-23}$ J/K the Boltzmann constant,

$$\hat{H} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_k \, \hat{N}_{\mathbf{k}s} \ , \tag{2}$$

is the quantum Hamiltonian without the zero-point energy,

$$\hat{N} = \sum_{\mathbf{k}} \sum_{s} \hat{N}_{\mathbf{k}s} \tag{3}$$

is the total number operator, and μ is the chemical potential, fixed by the conservation of the particle number. Here $\hat{N}_{\mathbf{k}s}$ is the number operator of photons with wavevector $\mathbf{k}$ and polarization $s = 1, 2$.

Gas of photons at thermal equilibrium (II)

Quantum statistical mechanics dictates that the thermal average of any operator $\hat{A}$ is obtained as

$$\langle \hat{A} \rangle_T = \frac{1}{\mathcal{Z}} Tr[\hat{A}\, e^{-\beta(\hat{H}-\mu\hat{N})}] \,. \tag{4}$$

In our case the calculations are simplified because $\mu = 0$. Let us suppose that $\hat{A} = \hat{H}$, it is then quite easy to show that

$$\langle \hat{H} \rangle_T = \sum_{\mathbf{k}} \sum_{s} \frac{\hbar\omega_k}{e^{\beta\hbar\omega_k} - 1} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_k \, \langle \hat{N}_{\mathbf{k}s} \rangle_T \,, \tag{5}$$

where

$$\bar{N}_k = \langle \hat{N}_{\mathbf{k}s} \rangle_T = \frac{1}{e^{\beta\hbar\omega_k} - 1} = \frac{1}{e^{\hbar c k/(k_B T)} - 1} \,. \tag{6}$$

Remember that $\hat{N}_{\mathbf{k}s} = \hat{a}^+_{\mathbf{k}s}\hat{a}_{\mathbf{k}s}$ with $\hat{a}^+_{\mathbf{k}s}$ and $\hat{a}_{\mathbf{k}s}$ the ladder operators, which create and annihlate photons with wavevector **k** and polarization s.

Gas of photons at thermal equilibrium (III)

In the continuum limit, where

$$\sum_{\mathbf{k}} \to L^3 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \,, \tag{7}$$

with L^3 the volume of a cubic box of size L, and taking into account that $\omega_k = ck$, one can write the thermal-averaged energy density

$$\bar{\mathcal{E}} = \frac{\langle \hat{H} \rangle_T}{L^3} \tag{8}$$

as

$$\bar{\mathcal{E}} = 2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{c\hbar k}{e^{\beta c\hbar k} - 1} = \frac{c\hbar}{\pi^2} \int_0^\infty dk \frac{k^3}{e^{\beta c\hbar k} - 1} \,, \tag{9}$$

where the factor 2 is due to the two possible polarizations ($s = 1, 2$).

Gas of photons at thermal equilibrium (IV)

By using $\omega = ck$ instead of k as integration variable one gets

$$\bar{\mathcal{E}} = \frac{\hbar}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1} = \int_0^\infty d\omega \ \rho(\omega) \ , \tag{10}$$

where

$$\rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \tag{11}$$

is the energy density per frequency, i.e. the familiar formula of the black-body radiation, obtained for the first time in 1900 by Max Planck.[1] The previous integral can be explicitly calculated and it gives

$$\bar{\mathcal{E}} = \frac{\pi^2 k_B^4}{15 c^3 \hbar^3} T^4 \ , \tag{12}$$

which is nothing but the Stefan-Boltzmann law.

[1] M. Planck, Ann. Physik, **306**, 719 (1900).

Gas of photons at thermal equilibrium (V)

In an similar way one determines the average number density of photons:

$$
\begin{aligned}
\bar{n} &= \frac{\langle \hat{N} \rangle_T}{L^3} = \frac{1}{\pi^2} \int_0^\infty dk\, k^2 \frac{1}{e^{\frac{\hbar c k}{k_B T}} - 1} \\
&= \frac{1}{\pi^2 c^3} \int_0^\infty d\omega\, \frac{\omega^2}{e^{\beta \hbar \omega} - 1} = \frac{2\zeta(3) k_B^3}{\pi^2 c^3 \hbar^3} T^3 \,. \qquad (13)
\end{aligned}
$$

where $\zeta(x)$ is the Riemann zeta function and $\zeta(3) \simeq 1.202$.
Notice that both energy density $\mathcal{E}$ and number density n of photons go to zero as the temperature T goes to zero.
We stress that our results are obtained at thermal equilibrium and under the condition of a vanishing chemical potential, meaning that the number of photons is not conserved when the temperature is varied.

Field operators for photons (I)

We introduce[2] the annihilation (absorption) vector field operator of photons as

$$\hat{\boldsymbol{V}}(\mathbf{r},t) = \sum_{\mathbf{k}s} \hat{a}_{\mathbf{k}s} \frac{e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}}{\sqrt{L^3}} \mathbf{u}_{\mathbf{k}s} \ , \tag{14}$$

which destroys a photon localized at the position $\mathbf{r}$ at time t. Here L^3 is the volume and $\mathbf{u}_{\mathbf{k}s}$ is the unit polarization vector. The corresponding creation vector field operator reads

$$\hat{\boldsymbol{V}}^{+}(\mathbf{r},t) = \sum_{\mathbf{k}s} \hat{a}^{+}_{\mathbf{k}s} \frac{e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}}{\sqrt{L^3}} \mathbf{u}_{\mathbf{k}s} \ . \tag{15}$$

The definition of the number density operator of photons follows quite naturally

$$\hat{n}(\mathbf{r},t) = \hat{\boldsymbol{V}}^{+}(\mathbf{r},t) \cdot \hat{\boldsymbol{V}}(\mathbf{r},t) \ . \tag{16}$$

---

[2] Mandel and Wolf, Optical Coherence and Quantum Optics, Chapters 12 and 13 (Cambridge Univ. Press, 1995).

Field operators for photons (II)

We find immediately that

$$\hat{n}(\mathbf{r},t) = \sum_{\mathbf{k}'s'} \sum_{\mathbf{k}s} \hat{a}^+_{\mathbf{k}'s'} \hat{a}_{\mathbf{k}s} \, \frac{e^{-i(\mathbf{k}'\cdot\mathbf{r}-\omega_{k'}t)} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}}{L^3} \, \mathbf{u}_{\mathbf{k}'s'} \cdot \mathbf{u}_{\mathbf{k}s} \ . \qquad (17)$$

Moreover, taking into account that the ortho-normalization of the plane waves we get

$$\int \hat{n}(\mathbf{r},t) \; d^3\mathbf{r} = \sum_{\mathbf{k}s} \hat{a}^+_{\mathbf{k}s} \hat{a}_{\mathbf{k}s} = \sum_{\mathbf{k}s} \hat{N}_{\mathbf{k}s} = \hat{N} \ . \qquad (18)$$

Thus, this operator is indeed the local number density operator of photons.

Field operators for photons (III)

By using the Hamiltonian of Eq. (2), the thermal average of the number density operator is given by

$$\langle \hat{n}(\mathbf{r}, t) \rangle_T = \frac{1}{L^3} \sum_{\mathbf{k}s} \langle \hat{a}^+_{\mathbf{k}s} \hat{a}_{\mathbf{k}s} \rangle_T = \frac{1}{L^3} \sum_{\mathbf{k}s} \langle \hat{N}_{\mathbf{k}s} \rangle_T = \frac{\langle \hat{N} \rangle_T}{L^3} \tag{19}$$

due to the fact that

$$\langle \hat{a}^+_{\mathbf{k}'s'} \hat{a}_{\mathbf{k}s} \rangle_T = \langle \hat{a}^+_{\mathbf{k}s} \hat{a}_{\mathbf{k}s} \rangle_T \ \delta_{\mathbf{k},\mathbf{k}'} \delta_{s,s'} \ . \tag{20}$$

First order correlations (I)

In several applications it is useful the field-field correlator given by

$$\langle \hat{\mathbf{V}}^{+}(\mathbf{r},t)\cdot\hat{\mathbf{V}}(\mathbf{r}',t')\rangle_T = \sum_{\mathbf{k}'s'}\sum_{\mathbf{k}s}\langle \hat{a}^{+}_{\mathbf{k}'s'}\hat{a}_{\mathbf{k}s}\rangle_T \, \frac{e^{-i(\mathbf{k}'\cdot\mathbf{r}-\omega_{k'}t)}e^{i(\mathbf{k}\cdot\mathbf{r}'-\omega_k t')}}{L^3}\, \mathbf{u}_{\mathbf{k}'s'}\cdot\mathbf{u}_{\mathbf{k}s} \; . \tag{21}$$

Using Eq. (20) we get

$$\langle \hat{\mathbf{V}}^{+}(\mathbf{r},t)\cdot\hat{\mathbf{V}}(\mathbf{r}',t')\rangle_T = \frac{1}{L^3}\sum_{\mathbf{k}s}\langle \hat{N}_{\mathbf{k}s}\rangle_T \; e^{i[\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})-\omega_k(t'-t)]} \; . \tag{22}$$

We can also normalize to one this field-field correlator introducing the so-called one-body correlation function

$$\begin{aligned} g^{(1)}(\mathbf{r}-\mathbf{r}',t-t') &= \frac{L^3}{\langle \hat{N}\rangle_T}\langle \hat{\mathbf{V}}^{+}(\mathbf{r},t)\cdot\hat{\mathbf{V}}(\mathbf{r}',t')\rangle_T \\ &= \frac{1}{\langle \hat{N}\rangle_T}\sum_{\mathbf{k}s}\langle \hat{N}_{\mathbf{k}s}\rangle_T \; e^{i[\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})-\omega_k(t'-t)]} \; . \end{aligned} \tag{23}$$

First order correlations (II)

Clearly, one finds

$$g^{(1)}(\mathbf{0}, 0) = 1 \ . \tag{24}$$

In the continuum limit where $\sum_{\mathbf{k}} \to L^3 \int d^3\mathbf{k}/(2\pi)^3$ and using spherical coordinates for the wavevector $\mathbf{k}$, the stationary one-body correlation function (23) can be written as

$$\begin{aligned} g^{(1)}(\mathbf{r}, t) &= \frac{4\pi L^3}{\langle \hat{N} \rangle_T} \int_0^{+\infty} \frac{dk \ k^2}{8\pi^3} \ \bar{N}_k \int_{-1}^{1} d[\cos(\theta)] \ e^{ik|\mathbf{r}|\cos\theta} \ e^{-ickt} \\ &= \frac{1}{\bar{n}} \frac{1}{\pi^2} \int_0^{+\infty} dk \ k^2 \ \bar{N}_k \frac{\sin(k|\mathbf{r}|)}{k|\mathbf{r}|} \ e^{-ickt} \ , \end{aligned} \tag{25}$$

where $\bar{n} = \langle \hat{N} \rangle / L^3$ and

$$\bar{N}_k = \frac{1}{e^{\hbar ck/(k_B T)} - 1} \ . \tag{26}$$

First order correlations (III)

In a more compact form the function $g^{(1)}(\mathbf{r}, t)$ reads

$$g^{(1)}(\mathbf{r}, t) = \frac{1}{2\zeta(3)} \int_0^{+\infty} dy \; y^2 \; \frac{1}{e^y - 1} \; \frac{\sin\left(y \frac{|\mathbf{r}|}{r_{ch}}\right)}{\left(y \frac{|\mathbf{r}|}{r_{ch}}\right)} \, e^{-iyt/t_{ch}} \; , \tag{27}$$

where

$$r_{ch} = \frac{\hbar c}{k_B T} \; , \tag{28}$$

is a characteristic length of the spatial decay and

$$t_{ch} = \frac{r_{ch}}{c} = \frac{\hbar}{k_B T} \tag{29}$$

is a characteristic time.

First order correlations (IV)

Considering only the time dependence, i.e. setting $\mathbf{r} = \mathbf{0}$, we have

$$g^{(1)}(\mathbf{0}, t) = \frac{1}{2\zeta(3)} \int_0^{+\infty} dy\ y^2\ \frac{1}{e^y - 1}\, e^{-iyt/t_{ch}} = \frac{\zeta(3, 1 + i(t/t_{ch}))}{\zeta(3)}\ , \tag{30}$$

where $\zeta(x, y)$ is the generalized Riemann zeta function defined as the analytic continuation of

$$\zeta(x, y) = \sum_{n=0}^{\infty} \frac{1}{(n + y)^x} \tag{31}$$

and such that $\zeta(x, 1) = \zeta(x)$.

Second order correlations (I)

The density-density correlator is defined as

$$\langle \hat{n}(\mathbf{r},t)\ \hat{n}(\mathbf{r}',t')\rangle_T = \sum_{\mathbf{k}'''s'''}\sum_{\mathbf{k}''s''}\sum_{\mathbf{k}'s'}\sum_{\mathbf{k}s}\langle \hat{a}^{+}_{\mathbf{k}'''s'''}\hat{a}_{\mathbf{k}''s''}\hat{a}^{+}_{\mathbf{k}'s'}\hat{a}_{\mathbf{k}s}\rangle_T$$
$$\times\quad \frac{e^{-i(\mathbf{k}'''\cdot\mathbf{r}-\omega_{k'''}t)}e^{i(\mathbf{k}''\cdot\mathbf{r}-\omega_{k''}t)}}{L^3}\,\frac{e^{-i(\mathbf{k}'\cdot\mathbf{r}'-\omega_{k'}t')}e^{i(\mathbf{k}\cdot\mathbf{r}'-\omega_{k}t')}}{L^3}$$
$$\times\quad \mathbf{u}_{\mathbf{k}'''s'''}\cdot\mathbf{u}_{\mathbf{k}''s''}\ \mathbf{u}_{\mathbf{k}'s'}\cdot\mathbf{u}_{\mathbf{k}s}\ . \qquad (32)$$

However, taking into account the Wick's theorem[3] Eq. (32) becomes

$$\langle \hat{n}(\mathbf{r},t)\ \hat{n}(\mathbf{r}',t')\rangle_T = \frac{1}{L^6}\sum_{\mathbf{k}'s'}\sum_{\mathbf{k}s}\langle \hat{N}_{\mathbf{k}'s'}\rangle_T\langle \hat{N}_{\mathbf{k}s}\rangle_T$$
$$\times\quad \left[1+e^{i[(\mathbf{k}'-\mathbf{k})\cdot(\mathbf{r}-\mathbf{r}')-(\omega_{k'}-\omega_k)(t-t')]}\ \varepsilon_{\mathbf{k}'s'}\cdot\varepsilon_{\mathbf{k}s}\right]\ . \qquad (33)$$

[3] MSc thesis of A. Hoffmann, Spatiotemporal formation of the Kondo cloud (Ludwig Maximilians Universität München, 2012).

Second order correlations (II)

We now introduce the normalized two-body correlation function

$$g^{(2)}(\mathbf{r}-\mathbf{r}', t-t') = \frac{L^6}{\langle \hat{N} \rangle^2} \langle \hat{n}(\mathbf{r}, t)\ \hat{n}(\mathbf{r}', t') \rangle_T \tag{34}$$

Quite remarkably, it is possibile to prove[4] that

$$g^{(2)}(\mathbf{r}, t) = 1 + |g^{(1)}(\mathbf{r}, t)|^2 \ . \tag{35}$$

Thus, the two-body correlation function can be obtained from the knowledge of the one-body correlation function. Clearly, one finds

$$g^{(2)}(\mathbf{0}, 0) = 2 \ . \tag{36}$$

[4] I. Bouchoule, N. J. Van Druten, and C.I. Westbrook, Atom chips and one-dimensional Bose gases, in J. Reichel and V. Vuletic (Eds.), Atom Chips, Chapter 11 (Wiley, 2011).

Second order correlations (III)

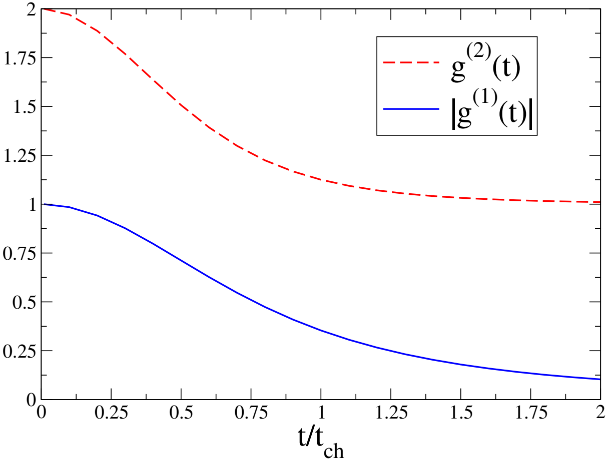

Figure: Time dependence of the one-body $|g^{(1)}(\mathbf{0}, t)|$ and two-body $g^{(2)}(\mathbf{0}, t)$ correlation functions. Here t is time and $t_{ch} = \hbar/(kB_T)$ is the characteristic period of time decay.

Detecting time correlations (I)

Considering the temperature

$$T = 2.73 \text{ Kelvin} \tag{37}$$

of the cosmic microwave background (CMB) we have

$$r_{ch} = \frac{\hbar c}{k_B T} = 8.4 \times 10^{-4} \text{ meters} , \tag{38}$$

and

$$t_{ch} = \frac{r_{ch}}{c} = \frac{\hbar}{k_B T} = 2.8 \times 10^{-12} \text{ seconds} . \tag{39}$$

Detecting time correlations (II)

Detectors of photons work in a finite range of linear frequencies

$$\nu \in [\nu_{min}, \nu_{max}] \qquad (40)$$

and within a specific solid angle Ω such that

$$\Omega \leq 4\pi \ . \qquad (41)$$

Thus, the detectable total number density of photons is

$$\bar{n}_d = 2\Omega \left(\frac{k_B T}{hc} \right)^3 \int_{h\nu_{min}/(k_B T)}^{h\nu_{max}/(k_B T)} dy \, \frac{y^2}{e^y - 1} \qquad (42)$$

with $\nu = \omega/(2\pi)$ and $h = 2\pi\hbar$. Similary, the detectable first order correlation function is

$$g_d^{(1)}(\mathbf{0}, t) = \frac{2\Omega}{\bar{n}_d} \left(\frac{k_B T}{hc} \right)^3 \int_{h\nu_{min}/(k_B T)}^{h\nu_{max}/(k_B T)} dy \, \frac{y^2}{e^y - 1} e^{-iyt/t_{ch}} \ . \qquad (43)$$

Detecting time correlations (III)

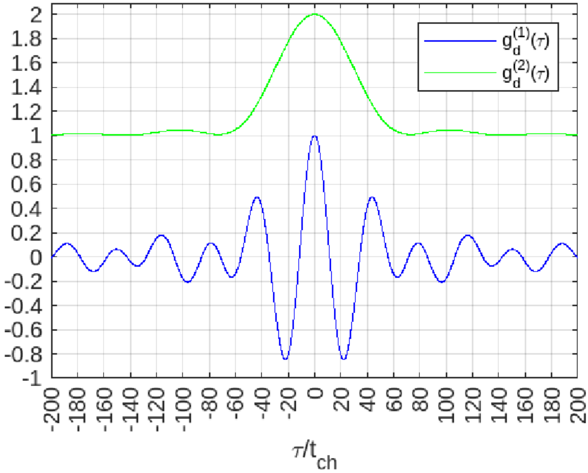

Figure: Correlations $|g_d^{(1)}(\mathbf{0}, t)|$ and $g_d^{(2)}(\mathbf{0}, t) = 1 + |g_d^{(1)}(\mathbf{0}, t)|^2$ vs t/t_{ch} with $t_{ch} = \hbar/(kB_T)$. Temperature $T = 2.73$ Kelvin and frequency range $[5, 10]$ GHz. Adapted from M. Toffoli, BSc thesis (2023).

Conclusions

- We have discussed some properties of photons at thermal equilibrium.
- First order and second order correlations are derived. The obtained formulas are not new.
- We have also considered a modification of these formulas taking into account detection limitations.
- Quite remarkably, the temporal correlations are strongly modified working with a finite range of photon frequencies.

Acknowledgements

Thank you for your attention!