Thermodynamics and sound modes of the unitary Fermi superfluid

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Summary

- Brief historical introduction
- Unitary Fermi gas
- Single-particle and collective excitations
- Universal thermodynamics
- Superfluid fraction and critical temperature
- First and second sound
- Density perturbation and sound mixing
- Conclusions

Brief historical introduction (I)

In 1924 Wolfgang Pauli introduced the concept of spin. Now we know that any particle has an intrinsic angular momentum, called spin $\vec{S} = (S_x, S_y, S_z)$, characterized by two quantum numbers s ed m_s, where for s fixed one has $m_s = -s, -s+1, ..., s-1, s$, and in addition

$$S_z = m_s \hbar \ ,$$

with $\hbar$ ($1.054 \cdot 10^{-34}$ Joule$\times$seconds) the reduced Planck constant.
In honour of Satyendra Nath Bose and Enrico Fermi all the particles are now divided into two groups:
– **bosons**, characterized by an integer s:

$$s = 0, 1, 2, 3, ...$$

– **fermions**, characterized by a half-integer s:

$$s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, ...$$

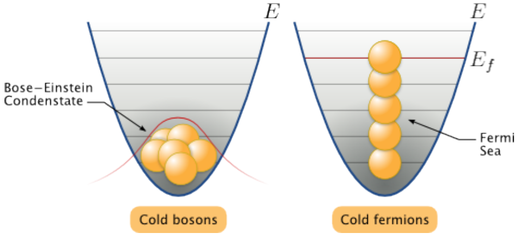
Examples: the photon is a boson ($s = 1$, $m_s = -1, 1$), while the electron is a fermion ($s = \frac{1}{2}$, $m_s = -\frac{1}{2}, \frac{1}{2}$).
Among "not elementary particles": helium ${}^4_2$He is a boson ($s = 0$, $m_s = 0$), while helium ${}^3_2$He is a fermion ($s = \frac{1}{2}$, $m_s = -\frac{1}{2}, \frac{1}{2}$).

Brief historical introduction (II)

A fundamental experimental and theoretical[1] result: identical bosons and identical fermions have a very different behavior!!

– Identical bosons can occupy the same single-particle quantum state, i.e. thay can stay together; if all bosons are in the same single-particle quantum state one has Bose-Einstein condensation.

– Identical fermions CANNOT occupy the same single-particle quantum state, i.e. they somehow repel each other: Pauli exclusion principle.



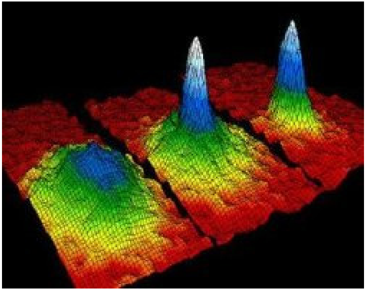
Identical bosons (a) and identical spin-polarized fermions (b) in a harmonic trap at very low temperature.

[1] Spin-statistics theorem [Markus Friez 1939; Wolfgang Pauli 1940].

Brief historical introduction (III)

In 1995 Eric Cornell, Carl Wieman e Wolfgang Ketterle [Nobel Prize in Physics 2001] achieved Bose-Einstein condensation (BEC) cooling gases of ^{87}Rb and ^{23}Na atoms.

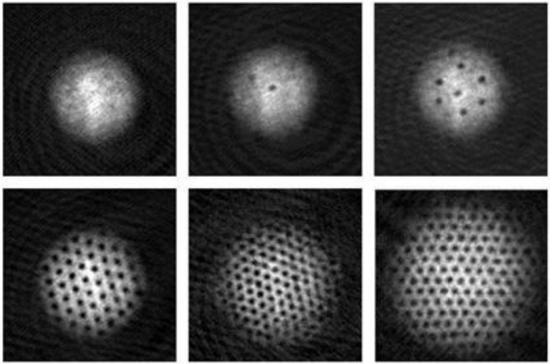
For these bosonic systems, which are very dilute and ultracold, the critical temperature to reach the BEC is about $T_{BEC} \simeq 100$ nanoKelvin.



Density profiles of a gas of Rubidium atoms: formation of the Bose-Einstein condensate. For an atom of ^{87}Rb the total nuclear spin is $I = \frac{3}{2}$, the total electronic spin is $S = \frac{1}{2}$, and the total atomic spin is $F = 1$ o $F = 2$: the neutral ^{87}Rb atom is a boson.

Brief historical introduction (IV)

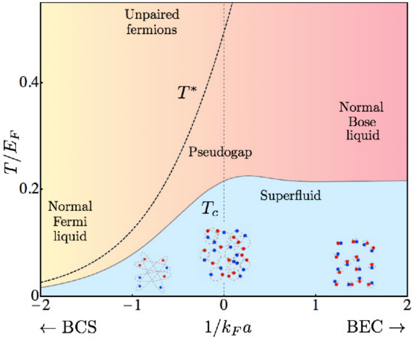
An interesting consequence of Bose-Einstein condensation with ultracold atoms is the possibility to generate quantized vortices: the system is **superfluid**!



Formation of quantized vortices in a condensed gas of ^{87}Rb atoms. The number of vortices increases by increasing the rotational frequency of the system.

Unitary Fermi gas (I)

In 2004 the 3D BCS-BEC crossover has been observed with **ultracold gases of two-component fermionic ^{40}K or ^{6}Li atoms.**[2]



This crossover is obtained by using a Fano-Feshbach resonance to change the strength of the effective inter-atomic attraction and, consequently, the 3D s-wave scattering length a. **Unitary Fermi gas**: $a \to \pm\infty$.

[2] C.A. Regal et al., PRL **92**, 040403 (2004); M.W. Zwierlein et al., PRL **92**, 120403 (2004); J. Kinast et al., PRL **92**, 150402 (2004).

Unitary Fermi gas (II)

Let us consider a **gas of atomic fermions** with two equally-populated spin components: $n_\uparrow = n_\downarrow$. The system is dilute if the characteristic range r_e of the inter-atomic potential is much smaller than the average interparticle separation $d = n^{-1/3}$ with total number density $n = n_\uparrow + n_\downarrow$, namely

$$r_e \ll d \ . \tag{1}$$

The system is strongly-interacting if the scattering length a of the inter-atomic potential greatly exceeds the average interparticle separation $d = n^{-1/3}$, i.e.

$$d \ll |a| \ . \tag{2}$$

The unitarity regime[3] is characterized by both these conditions:

$$r_e \ll d \ll |a| \ . \tag{3}$$

Under these conditions the dilute but strongly-interacting Fermi gas is called **unitary Fermi gas**.

[3] S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 80, 1215 (2008).

Unitary Fermi gas (III)

Ideally, the unitarity limit corresponds to

$$r_e = 0 \qquad \text{and} \qquad a = \pm\infty \; . \tag{4}$$

In a uniform configuration and at zero temperature, the only length characterizing the Fermi gas in the unitarity limit is the average interparticle distance $d = n^{-1/3}$.
In this case the ground-state energy must be[4]

$$E_{gs} = \xi \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n^{2/3} N = \xi \frac{3}{5} \epsilon_F N \tag{5}$$

with $\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2)^{2/3} n^{2/3}$ Fermi energy of the ideal gas and ξ a universal unknown parameter: Bertsch parameter.
Monte Carlo calculations and experimental data with dilute and ultracold atoms suggest that, at zero temperature, the unitary Fermi gas is a superfuid with $\xi \simeq 0.4$.

[4] W. Zwerger (Ed.), The BCS-BEC Crossover and the Unitary Fermi Gas (Springer, 2011). Simply for dimensional reasons.

Single-particle and collective excitations (I)

Inspired by the Landau theory of elementary excitations we model the many-body quantum Hamiltonian $\hat{H}$ of the uniform unitary Fermi gas with the simple effective Hamiltonian

$$\hat{H} = E_{gs} + \sum_{\sigma=\uparrow,\downarrow} \sum_{\mathbf{k}} \epsilon_{\mathrm{sp}}(k) \; \hat{c}^{\dagger}_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} \epsilon_{\mathrm{col}}(q) \; \hat{b}^{\dagger}_{\mathbf{q}} \hat{b}_{\mathbf{q}} \; , \qquad (6)$$

where
the $\hat{c}^{\dagger}_{\mathbf{k}\sigma}$ ($\hat{c}_{\mathbf{k}\sigma}$) operator creates (annihilates) a single-particle excitation, respectively, with linear momentum $\mathbf{k}$, spin σ, and energy $\epsilon_{\mathrm{sp}}(k)$,
whereas
the $\hat{b}^{\dagger}_{\mathbf{q}}$ ($\hat{b}_{\mathbf{q}}$) operator creates (annihilates) a bosonic collective excitation, respectively, of linear momentum $\mathbf{q}$ and energy $\epsilon_{\mathrm{col}}(q)$.

Single-particle and collective excitations (II)

The dispersion of the BCS-like single-particle elementary excitations can be written as

$$\epsilon_{\rm sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \zeta\epsilon_F\right)^2 + \Delta_0^2} \qquad (7)$$

where ζ is a parameter taking into account the interaction between fermions and the reconstruction of the Fermi surface close to the critical temperature. In particular, $\zeta = 0.9$ according[5] to accurate Monte Carlo results. Moreover, Δ_0 is the gap parameter, with $2\Delta_0$ the minimal energy to break a Cooper pair. The gap energy Δ_0 of the unitary Fermi gas at zero-temperature has been calculated with Monte Carlo simulations[6] and found to be

$$\Delta_0 = \gamma\epsilon_F \qquad (8)$$

with $\gamma = 0.45$.

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[5] P. Magierski, G. Wlazlowski, A. Bulgac, and J. E. Drut, Phys. Rev. Lett. **103**, 210403 (2009).

[6] J. Carlson and S. Reddy, Phys. Rev. Lett. **95**, 060401 (2005).

Single-particle and collective excitations (III)

The dispersion relation of collective elementary excitations is assumed to be given by

$$\epsilon_{\text{col}}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(2mc_B^2 + \lambda \frac{\hbar^2 q^2}{2m}\right)} , \qquad (9)$$

where $c_B = \sqrt{\xi/3}\; v_F$ is the Bogoliubov sound velocity with $v_F = \sqrt{2\epsilon_F/m}$ the Fermi velocity of a non-interacting Fermi gas.
In a old paper [LS, Phys. Rev. A **82**, 063619 (2010)] we used the value $\lambda = 0.25$, which is consistent with a macroscopic time-dependent nonlinear Schrödinger equation approach without the inclusion of spurious terms.[7]
In a recent paper [G. Bighin, A. Cappellaro, and LS, Phys. Rev. A **105**, 063329 (2022)] we have used instead $\lambda = 0.08$, which is the value obtained[8] from the beyond-mean-field GPF theory[9] at unitarity.

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[7] LS and F. Toigo, Phys. Rev. A 78, 053626 (2010).
[8] G. Bighin, LS, P. A. Marchetti, and F. Toigo, Phys. Rev. A 92, 023638 (2015).
[9] J. Tempere and J. P. Devreese, Superconductors: Materials, Properties and Applications, InTech 383 (2012).

Universal thermodynamics (I)

In the canonical ensemble the **Helmholtz free energy** F of the system is obtained from the partition function $\mathcal{Z}$ as follows

$$\mathcal{Z} = \mathrm{Tr}[e^{-\hat{H}/(k_B T)}] = e^{-F/(k_B T)} \ . \qquad (10)$$

Similarly to Eq. (6), the free energy of the unitary Fermi gas can be written as

$$F = F_{gs} + F_{\rm sp} + F_{\rm col} \ , \qquad (11)$$

where F_{gs} is the free energy of the ground-state,

$$F_{\rm sp} = -2 k_B T \sum_{\mathbf{k}} \ln[1 + e^{-\epsilon_{\rm sp}(k)/(k_B T)}] \qquad (12)$$

is the free energy of fermionic single-particle excitations and finally

$$F_{\rm col} = -k_B T \sum_{\mathbf{q}} \ln[1 - e^{-\epsilon_{\rm col}(q)/(k_B T)}] \qquad (13)$$

is the free energy of the bosonic collective excitations.

Universal thermodynamics (II)

The total **Helmholtz free energy** F of a unitary Fermi gas in the superfluid phase can be then written[10] as

$$F = N\epsilon_F \Phi(x) \ , \qquad (14)$$

where, due to the scale-invariance of the system, $\Phi(x)$ is a function of the scaled temperature $x \equiv T/T_F$ only, having defined the Fermi temperature $T_F = \epsilon_F/k_B$. Explicitly, $\Phi(x)$ takes the following form

$$\begin{aligned} \Phi(x) &= \frac{3}{5}\xi - 3x \int_0^{+\infty} \ln\left[1 + e^{-\tilde{\epsilon}_{\mathrm{sp}}(u)/x}\right] u^2 \mathrm{d}u \\ &+ \frac{3}{2}x \int_0^{+\infty} \ln\left[1 - e^{-\tilde{\epsilon}_{\mathrm{col}}(u)/x}\right] u^2 \mathrm{d}u \ . \end{aligned} \qquad (15)$$

Note that the discrete summations have been replaced by integrals, and that we set $\tilde{\epsilon}_{\mathrm{col}}(u) = \sqrt{u^2(4\xi/3 + \lambda u^2)}$ and $\tilde{\epsilon}_{\mathrm{sp}}(u) = \sqrt{(u^2 - \zeta)^2 + \gamma^2}$.

[10] LS, Phys. Rev. A **82**, 063619 (2010).

Universal thermodynamics (III)

We now aim at calculating the thermodynamics of the system in terms of the universal function $\Phi(x)$ and its derivatives. From the **Helmholtz free energy** F we can immediately obtain the chemical potential μ, that is defined as

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = \epsilon_F \left[\frac{5}{3}\Phi\left(\frac{T}{T_F}\right) - \frac{2}{3}\frac{T}{T_F}\Phi'\left(\frac{T}{T_F}\right)\right] , \qquad (16)$$

where $\Phi'(x) = \frac{d\Phi(x)}{dx}$ and one recovers $\mu_0 = \xi\epsilon_F$ in the limit of zero-temperature.

The entropy S is readily calculated from the free energy F through the relation

$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V} = -Nk_B\Phi'(x) . \qquad (17)$$

where $\Phi'(x)$ is the first derivative of Φ with respect to x.

Universal thermodynamics (IV)

Furthermore, the internal energy $E = F + TS$, can immediately be rewritten as

$$E = N\epsilon_F \left[\Phi(x) - x\ \Phi'(x)\right] \tag{18}$$

and, similarly, the pressure P is related to the free energy F by the simple relation

$$P = -\left(\frac{\partial F}{\partial V}\right)_{N,T} = \frac{2}{3} n\epsilon_F \left[\Phi(x) - x\Phi'(x)\right] \ . \tag{19}$$

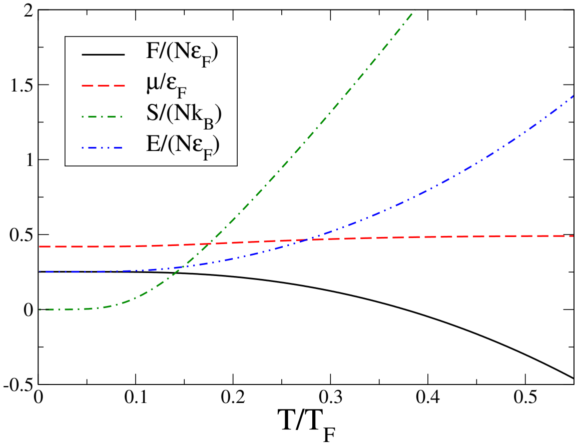
Remark: Adopting the Maxwell-Boltzmann distribution for fermionic single-particles instead of the Fermi-Dirac one, and under the further assumption that $\lambda = 0$, the adimensional fee energy becomes

$$\Phi(x) \simeq \frac{3}{5}\xi - \frac{\pi^4\sqrt{3}}{80\ \xi^{3/2}} x^4 - \frac{3\sqrt{2\pi}}{2} \zeta^{1/2} \gamma^{1/2} x^{3/2} e^{-\gamma/x} \ . \tag{20}$$

This expression was proposed by Bulgac, Drut and Magierski.[11] We call this equation the BDM model.

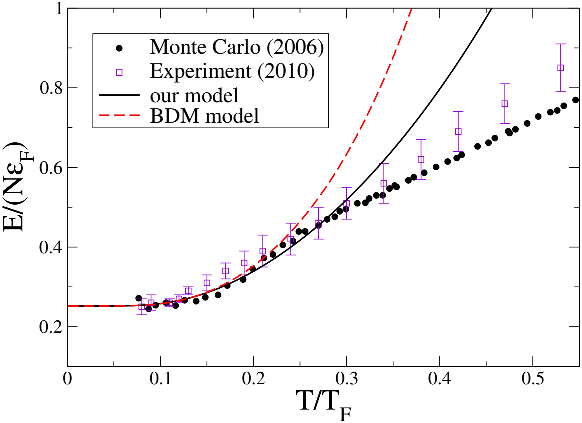
[11] A. Bulgac, J.E. Drut, and P. Magierski, Phys. Rev. Lett **96**, 090404 (2006).

Universal thermodynamics (V)



Thermodynamical quantities of the unitary Fermi gas deduced from our model as a function of the adimensional temperature T/T_F with $T_F = \epsilon_F/k_B$ the Fermi temperature. Plot taken from LS, Phys. Rev. A **82**, 063619 (2010), where $\xi = 0.42$, $\lambda = 0.25$, $\zeta = 0.9$, and $\gamma = 0.45$.

Universal thermodynamics (VI)



Scaled internal energy $E/(N\epsilon_F)$ as a function of the scaled temperature T/T_F. Filled circles: Monte Carlo simulations [Phys. Rev. A **78**, 023625 (2008)]. Open squares with error bars: experimental data [Science **442**, 327 (2010)]. Solid line: our model with $\xi = 0.42$, $\lambda = 0.25$, $\zeta = 0.9$, and $\gamma = 0.45$. Dashed line: Bulgac-Drut-Magierski (BDM) model. Plot taken from LS, Phys. Rev. A **82**, 063619 (2010).

Superfluid fraction and critical temperature (I)

For a **viscous fluid** with a quite generic zero-temperature bulk chemical potential $\mu(n)$ the equations of viscous hydrodynamics are given by

$$\begin{aligned} \frac{\partial}{\partial t} n + \boldsymbol{\nabla} \cdot (n\,\mathbf{v}) &= 0 \\ m \frac{\partial}{\partial t}\mathbf{v} + \boldsymbol{\nabla}\left[\frac{1}{2} m v^2 + \mu(n)\right] &= \eta \nabla^2 \mathbf{v} + m\mathbf{v} \wedge (\boldsymbol{\nabla} \wedge \mathbf{v}) \end{aligned}$$

where n is the local number density and $\mathbf{v}$ is the local velocity. Here η is the **viscosity** and a **rotational term** appears. These equations are called Navier-Stokes equations.

One obtains the zero-temperature equations of superfluid hydrodynamics, where the local number density n coincides with the superfluid density n_s and the local velocity $\mathbf{v}$ coincides with the superfluid velocity v_s, setting

$$\eta = 0\,, \tag{21}$$

$$\boldsymbol{\nabla} \wedge \mathbf{v} = \mathbf{0}\,. \tag{22}$$

Superfluid fraction and critical temperature (II)

Thus, the $T = 0$ equations of superfluid hydrodynamics are given by

$$\begin{aligned} \frac{\partial}{\partial t} n_s + \nabla \cdot (n_s \mathbf{v}_s) &= 0 \\ m \frac{\partial}{\partial t} \mathbf{v}_s + \nabla \left[\frac{1}{2} m v_s^2 + \mu(n_s) \right] &= \mathbf{0} \, . \end{aligned}$$

Within the superfluid hydrodynamics quantum effects are encoded not only in the equation of state, i.e. $\mu = \mu(n_s)$, but also into the properties of the local velocity field $\mathbf{v}_s(\mathbf{r}, t)$: it is proportional to the gradient of a scalar field, $\theta(\mathbf{r}, t)$, that is the angle of the phase of a single-valued complex wavefunction $\psi(\mathbf{r}, t)$.
In other words, $\mathbf{v}_s(\mathbf{r}, t)$ must satisfy the equation

$$\oint_{\mathcal{C}} \mathbf{v}_s \cdot d\mathbf{r} = \frac{\hbar}{m} \oint_{\mathcal{C}} \nabla\theta \cdot d\mathbf{r} = \frac{\hbar}{m} \oint_{\mathcal{C}} d\theta = \frac{\hbar}{m} 2\pi \, k \qquad (23)$$

for any closed contour $\mathcal{C}$, with k an integer number. The circulation is quantized in units of $\hbar/m$, and this property is strictly related to the existence of quantized vortices.

Superfluid fraction and critical temperature (III)

At finite temperature T, the equations of superfluid hydrodynamics are much more complicated because they involve several fields: the superfluid density n_s, the normal density n_n, the superfluid velocity $\mathbf{v}_s$ and the normal velocity $\mathbf{v}_n$.
According to Landau's two-fluid theory[12] the total number density n of a system in the superfluid phase can be written as

$$n = n_{\mathrm{s}} + n_{\mathrm{n}} \; , \tag{24}$$

where n_{s} is the superfluid density and n_{n} is the **normal density**.
Naturally, at zero temperature the whole system is in the superfluid phase, and one has $n_{\mathrm{n}} = 0$ and $n = n_{\mathrm{s}}$. As the temperature T increases, the normal density n_{n} increases, as well, until at the critical temperature T_c one has $n_n = n$ and, correspondingly, $n_s = 0$.

[12] L.D. Landau, J. Phys. (USSR) **5**, 71 (1941).

Superfluid fraction and critical temperature (IV)

Within our scheme, the normal density of a unitary gas is given the sum of two contributions

$$n_{\mathrm{n}} = n_{\mathrm{n,sp}} + n_{\mathrm{n,sp}} \;, \tag{25}$$

i.e. a contribution $n_{\mathrm{n,sp}}$ from to the single-particle excitations and a contribution $n_{\mathrm{n,col}}$ from collective excitations.
At thermal equilibrium, Landau was able to connect[13] the normal densities to their quantum statistics and to their energy spectrum.
In the present case the single-particle contribution to the normal density reads

$$n_{\mathrm{n,sp}} = \frac{1}{k_B T V} \sum_{\mathbf{k}} k^2 \frac{e^{\epsilon_{\mathrm{sp}}(k)/(k_B T)}}{(e^{\epsilon_{\mathrm{sp}}(k)/(k_B T)} + 1)^2} \;, \tag{26}$$

whereas, concerning the contribution from the collective modes,

$$n_{\mathrm{n,col}} = \frac{1}{2 k_B T V} \sum_{\mathbf{q}} q^2 \frac{e^{\epsilon_{\mathrm{col}}(q)/(k_B T)}}{(e^{\epsilon_{\mathrm{col}}(q)/(k_B T)} - 1)^2} \;. \tag{27}$$

[13] L.D. Landau, J. Phys. (USSR) **5**, 71 (1941).

Superfluid fraction and critical temperature (V)

It is then easy to derive the superfluid fraction

$$\frac{n_s}{n} = 1 - \mathcal{N}(x) \; , \tag{28}$$

where the universal function $\mathcal{N}(x)$ of the normal fraction is again a function of the scaled temperature $x \equiv T/T_F$ only, explicitly given by

$$\begin{aligned} \mathcal{N}(x) &= \frac{2}{x} \int_0^{+\infty} \frac{e^{\tilde{\epsilon}_{sp}(\eta)/x}}{(e^{\tilde{\epsilon}_{sp}(\eta)/x} + 1)^2} \; \eta^4 \mathrm{d}\eta \\ &+ \frac{1}{x} \int_0^{+\infty} \frac{e^{\tilde{\epsilon}_{col}(\eta)/x}}{(e^{\tilde{\epsilon}_{col}(\eta)/x} - 1)^2} \; \eta^4 \mathrm{d}\eta \; , \end{aligned} \tag{29}$$

where we have converted sums to integrals.

Superfluid fraction and critical temperature (VI)

The superfluid density defines the **critical temperature** T_c of the superfluid-to-normal phase transition via the condition $n_s = 0$. With our choice of parameters for the elementary excitations we find

$$T_c = 0.23 \ T_F \ . \tag{30}$$

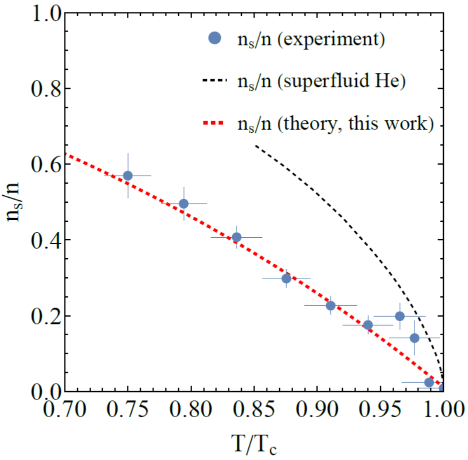
It must be pointed out that, while this estimation of the critical temperature agrees with more refined approaches, such as the functional GPF theory[14] or the NSR scheme,[15] it actually differs from the most recent experimental results,[16] placing it at $T_c/T_F \simeq 0.17$.
The overestimation of our theoretical critical temperature with respect to the experimental ones does not appear plotting the physical quantities vs T/T_c.

[14] H. Hu, X. J. Liu, and P. D. Drummond, EPL **74**, 574 (2007); J. Tempere and J. P. Devreese, Superconductors: Materials, Properties and Applications, InTech 383 (2012).

[15] P. Nozieres and S. Schmitt-Rink, J. Low. Temp. Phys. 59, 195 (1985).

[16] X. Li *et al.*, Science **375**, 528 (2022).

Superfluid fraction and critical temperature (VII)



Superfluid fraction n_s/n as a function of the adimensional temperature T/T_c. Comparison between our theory and recent experimental data [X. Li *et al.*, Science **375**, 528 (2022)]. Plot adapted from G. Bighin, A. Cappellaro, and LS, Phys. Rev. A **105**, 063329 (2022), where $\xi = 0.38$, $\lambda = 0.08$, $\zeta = 0.9$, and $\gamma = 0.45$.

First and second sound (I)

According to Landau's two-fluid equations, in a superfluid a local perturbation excites two wave-like modes, the first and the second sound, which propagate with velocities u_1 and u_2. These velocities are determined by the positive solutions of the algebraic biquadratic equation

$$u^4 + (c_{10}^2 + c_{20}^2)u^2 + c_T^2 c_{20}^2 = 0 \ , \tag{31}$$

where

$$c_{10} = \sqrt{\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_{\bar{S},V}} = v_F\sqrt{\frac{5}{9}\Phi(x) - \frac{5}{9}\frac{T}{T_F}\Phi'(x)} \tag{32}$$

is the adiabatic sound velocity with $\bar{S} = S/N$ the entropy per particle,

$$c_{20} = \sqrt{\frac{1}{m}\frac{\bar{S}^2}{\left(\frac{\partial \bar{S}}{\partial T}\right)_{N,V}}\frac{n_s}{n_n}} = v_F\sqrt{-\frac{1}{2}\frac{\Phi'(x)^2}{\Phi''(x)}\frac{1-\Xi(x)}{\Xi(x)}} \tag{33}$$

is the entropic sound velocity, and

$$c_T = \sqrt{\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_{T,V}} = v_F\sqrt{\frac{5}{9}\left(\Phi(x) - \frac{T}{T_F}\Phi'(x)\right) + \frac{2}{9}x^2\Phi''(x)} \tag{34}$$

is the isothermal sound velocity.

First and second sound (II)

The first sound u_1 is the largest of the two positive roots of Eq. (31) while the second sound u_2 is the smallest positive one. Thus

$$u_{1,2} = \sqrt{\frac{c_{10}^2 + c_{20}^2}{2} \pm \sqrt{\left(\frac{c_{10}^2 + c_{20}^2}{2}\right)^2 - c_{20}^2 c_T^2}} \, . \qquad (35)$$

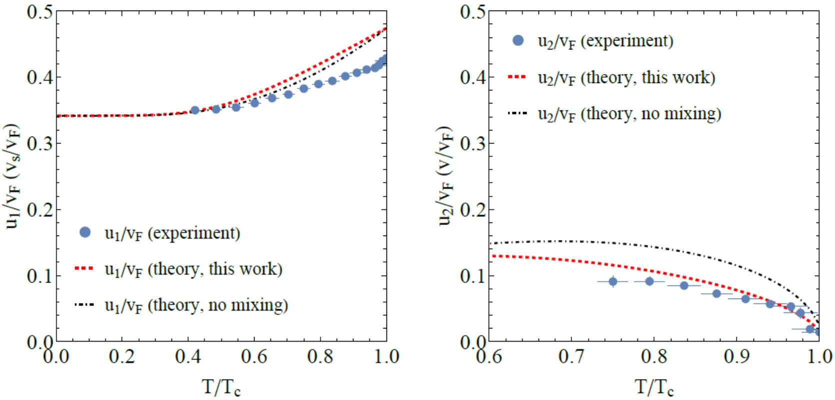
For the sake of completeness, we stress that the "Einstein-like relation"

$$\frac{E}{N} = \frac{10}{9} m c_{10}^2 \qquad (36)$$

derived in a recent paper[17] is automatically verified within our universal thermodynamic formalism, that naturally includes the scale-invariance of the unitary Fermi gas.

[17] P. B. Patel *et al.*, Science **370**, 1222 (2020).

First and second sound (III)



First sound velocity u_1 and second sound velocity u_2 as a function of the adimensional temperature T/T_c. Here $v_F = \sqrt{2\epsilon_F/m}$ is the Fermi velocity. Comparison between our theory and recent experimental data [X. Li *et al.*, Science **375**, 528 (2022)]. "No mixing" means the (wrong) assumption that $c_T = c_{10}$. Plot adapted from G. Bighin, A. Cappellaro, and LS, Phys. Rev. A **105**, 063329 (2022).

Density perturbation and sound mixing (I)

It is useful to analyze the first sound amplitude W_1 and the second sound amplitude W_2 of the response to a density perturbation, i.e.

$$\delta n(x,t) = W_1 \delta n_1(x \pm u_1 t) + W_2 \delta n_2(x \pm u_2 t) \tag{37}$$

In 2014 Ozawa and Stringari[18] obtained the following remarkable formulas for a generic superfluid

$$\frac{W_1}{W_1 + W_2} = \frac{(u_1^2 - c_{20}^2)\, u_2^2}{(u_1^2 - u_2^2)\, c_{20}^2} \tag{38}$$

and

$$\frac{W_2}{W_1 + W_2} = \frac{(c_{20}^2 - u_2^2)\, u_1^2}{(u_1^2 - u_2^2)\, c_{20}^2} \tag{39}$$

which give W_1 and W_2 in terms of three velocities: the first sound velocity u_1, the second sound velocity u_2, the entropic sound velocity c_{20}.

[18] T. Ozawa and S. Stringari, Phys. Rev. Lett. **112**, 025302 (2014).

Density perturbation and sound mixing (II)

Superfluid ^{4}He is characterized by "no mode mixing", i.e. $c_{10} \simeq c_T$: the first sound corresponds to a standard density wave with $u_1 \simeq c_{10} \simeq c_T$ and the second sound is understood as an entropy wave with $u_1 \simeq c_{20}$.

Thus, under the strict "no mode mixing" condition $c_{10} = c_T$, Eqs. (38) and (39) give $W_1 = 1$ and $W_2 = 0$. This means that, in this case, a density probe excites only the first sound mode.

For the unitary Fermi gas the situation is radically different[19] because the isothermal velocity c_T and the adiabatic velocity c_{10} are not always close each other. Indeed, we find that, in general, $W_1 \neq 1$ and $W_2 \neq 0$ for the unitary Fermi gas.

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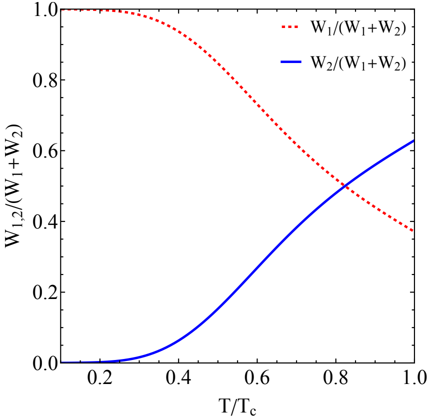
[19] L. P. Pitaevskii and S. Stringari, pp. 322-347, in Universal Themes of Bose-Einstein Condensation Edited by N.P. Proukakis, D.W. Snoke, and P.B. Littlewood (Cambridge University Press, 2017).

Density perturbation and sound mixing (III)



Contribution from the first (dashed red line) and second sound (**solid blue line**) to the amplitude of a density response as a function of the scaled temperature T/T_c. Figure adapted from G. Bighin, A. Cappellaro, and LS, Phys. Rev. A **105**, 063329 (2022).

Conclusions

- A simple description in terms of fermionic single-particle and bosonic collective elementary excitations is able to reproduce many properties of the **unitary Fermi gas**.
- The internal energy derived from our model is in good agreement with Monte Carlo simulations and experimental results for $T \leq 0.25 T_F$.
- We have reproduced the recently-measured superfluid fraction, first sound and second sound near the critical temperature $T_c \simeq 0.2 T_F$.
- Contrary to Helium 4, near the critical temperature the first and second sound of the **unitary Fermi gas** cannot be interpreted as a pure pressure-density wave and a pure entropy-temperature wave, respectively.
- Our investigation of the **unitary Fermi gas** shows that at very low temperatures the mixing of pressure-density and entropy-temperature oscillations is absent, whereas approaching T_c a density probe will excite both sounds.

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