Statistical Physics of Ultracold Atoms on the Surface of a Sphere

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Workshop/School “Frontiers in ultracold quantum gases”,
Menorca, June 10, 2026

Summary

- Historical introduction
- Statistical physics of trapped bosons
- Bose gas on the surface of a sphere
- Statistical physics of trapped fermions
- Fermi gas on the surface of a sphere
- Conclusions

Historical introduction (I)

In 1924 Wolfgang Pauli introduced the concept of spin. Now we know that any particle has an intrinsic angular momentum, called spin $\vec{S} = (S_x, S_y, S_z)$, characterized by two quantum numbers s ed m_s, where for s fixed one has $m_s = -s, -s+1, ..., s-1, s$, and in addition

$$S_z = m_s \hbar \ ,$$

with $\hbar$ ($1.054 \cdot 10^{-34}$ Joule$\times$seconds) the reduced Planck constant.
In honour of Satyendra Nath Bose and Enrico Fermi all the particles are now divided into two groups:
– **bosons**, characterized by an integer s:

$$s = 0, 1, 2, 3, ...$$

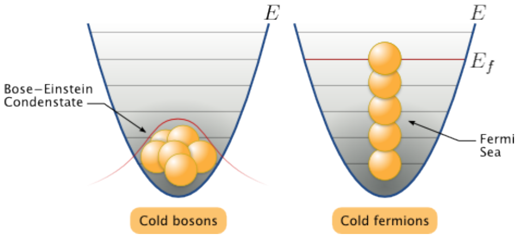
– **fermions**, characterized by a half-integer s:

$$s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, ...$$

Examples: the photon is a boson ($s = 1$, $m_s = -1, 1$), while the electron is a fermion ($s = \frac{1}{2}$, $m_s = -\frac{1}{2}, \frac{1}{2}$).
Among "not elementary particles": helium $^4_2$He is a boson ($s = 0$, $m_s = 0$), while helium $^3_2$He is a fermion ($s = \frac{1}{2}$, $m_s = -\frac{1}{2}, \frac{1}{2}$).

Historical introduction (II)

A fundamental experimental and theoretical[1] result: identical bosons and identical fermions have a very different behavior!!
– Identical bosons can occupy the same single-particle quantum state, i.e. they can stay together; if all bosons are in the same single-particle quantum state one has Bose-Einstein condensation.
– Identical fermions CANNOT occupy the same single-particle quantum state, i.e. they somehow repel each other: Pauli exclusion principle.

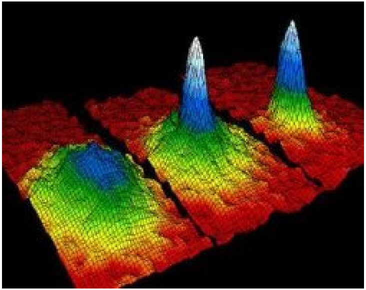


Identical bosons (a) and identical spin-polarized fermions (b) in a harmonic trap at very low temperature.

[1] Spin-statistics theorem [Markus Friez 1939; Wolfgang Pauli 1940].

Historical introduction (III)

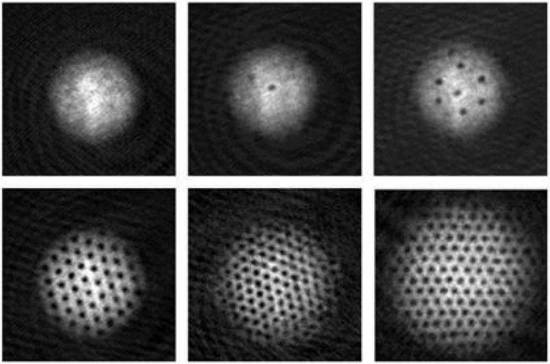
In 1995 Eric Cornell, Carl Wieman e Wolfgang Ketterle [Nobel Prize in Physics 2001] achieved Bose-Einstein condensation (BEC) cooling gases of ^{87}Rb and ^{23}Na atoms.
For these bosonic systems, which are very dilute and ultracold, the critical temperature to reach the BEC is about $T_{BEC} \simeq 100$ nanoKelvin.



Density profiles of a gas of Rubidium atoms: formation of the Bose-Einstein condensate. For an atom of ^{87}Rb the total nuclear spin is $I = \frac{3}{2}$, the total electronic spin is $S = \frac{1}{2}$, and the total atomic spin is $F = 1$ o $F = 2$: the neutral ^{87}Rb atom is a boson.

Historical introduction (IV)

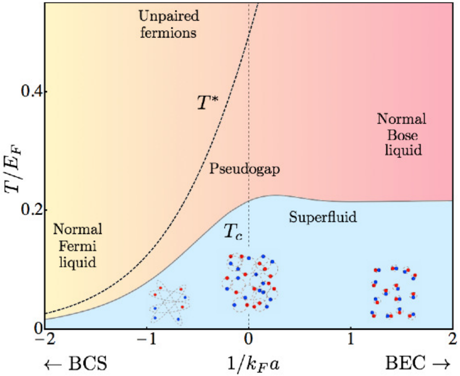
An interesting consequence of Bose-Einstein condensation with ultracold atoms is the possibility to generate quantized vortices: the system is **superfluid**!



Formation of quantized vortices in a condensed gas of ^{87}Rb atoms. The number of vortices increases by increasing the rotational frequency of the system.

Historical introduction (V)

In 2004 the 3D BCS-BEC crossover has been observed with **ultracold gases made of two-component fermionic ^{40}K or ^{6}Li atoms**.[2]



This crossover is obtained using a Fano-Feshbach resonance to change the 3D s-wave scattering length a_F of the inter-atomic potential.

[2] C.A. Regal et al., PRL **92**, 040403 (2004); M.W. Zwierlein et al., PRL **92**, 120403 (2004); J. Kinast et al., PRL **92**, 150402 (2004).

Historical introduction (VI)

In 2020 a BEC in harmonic trap[3] has been observed under microgravity with the NASA's Cold Atom Laboratory onboard of the International Space Station.

Moreover, in 2022 the same team has reported the observation of ultracold atomic bubbles[4] by using the bubble trap proposed by Zobay, Garraway and Perrin[5]

[3] D.C. Aveline et al., Nature **582**, 193 (2020).
[4] R.A. Carollo et al., Nature **606**, 281 (2022).
[5] O. Zobay and B. M. Garraway, Phys. Rev. Lett. **86**, 1195 (2001); B. M. Garraway and H. Perrin, J. Phys. B **49**, 172001 (2016).

Statistical physics of trapped bosons (I)

Let us consider a trapped gas of non-interacting spinless bosons, which can occupy the single-particle energy levels ϵ_α of the single-particle quantum states $|\alpha\rangle$, where α takes into account the single-particle quantum numbers and labels the single-particle quantum states as follows $\alpha = 0, 1, 2, 3, ...$
The internal energy is given by

$$E = \sum_\alpha \epsilon_\alpha \, N_\alpha \ , \tag{1}$$

where

$$N_\alpha = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1} \tag{2}$$

is the termal average of the number of bosons in the single-particle quantum state $|\alpha\rangle$. That is the Bose-Einstein distribution.
The thermal average of the total number of bosons then reads

$$N = \sum_\alpha N_\alpha = \sum_\alpha \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1} \ . \tag{3}$$

At fixed temperature $T = \frac{1}{k_B \beta}$, N is fully determined by the chemical potential μ.

Statistical physics of trapped bosons (II)

Let us suppose for simplicity that the set of single-particle energy levels ϵ_α is given by

$$\epsilon_0 \, , \; \epsilon_1 \, , \; \epsilon_2 \, , \; \epsilon_3 \; , \; \dots \tag{4}$$

where

$$\epsilon_0 < \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \; \dots \, . \tag{5}$$

Clearly it must be

$$\mu < \epsilon_0 \tag{6}$$

to avoid divergences in the Bose-Einstein distributions of each

$$N_\alpha = \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1} \, . \tag{7}$$

Moreover, as $\mu \to \epsilon_0$ the distribution

$$N_0 = \frac{1}{e^{\beta(\epsilon_0 - \mu)} - 1} \tag{8}$$

becomes very large: under this condition we have Bose-Einstein condensation, i.e. a macrosopic number of bosons in the lowest single-particle energy level ϵ_0.

Statistical physics of trapped bosons (III)

In the presence of Bose-Einstein condensation (BEC) it is useful to write the total number of bosons as follows

$$N = N_0 + \sum_{\alpha \neq 0} N_\alpha = \frac{1}{e^{\beta(\epsilon_0 - \mu)} - 1} + \sum_{\alpha \neq 0} \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1} \ . \tag{9}$$

The exact condensate fraction is defined as

$$\frac{N_0}{N} = 1 - \frac{\sum_{\alpha \neq 0} \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1}}{\sum_{\alpha} \frac{1}{e^{\beta(\epsilon_\alpha - \mu)} - 1}} \ . \tag{10}$$

Following Einstein (1924), instead of using the exact formula of N_0 we assume that N_0 is unknown but we also set

$$\mu = \epsilon_0 \tag{11}$$

in the BEC phase (i.e. when $N_0 > 0$). In this way, in the BEC phase we find

$$N = N_0 + \sum_{\alpha \neq 0} \frac{1}{e^{\beta(\epsilon_\alpha - \epsilon_0)} - 1} \ . \tag{12}$$

Statistical physics of trapped bosons (IV)

At the critical temperature T_{BEC} we have $N_0 = 0$ and consequently

$$N = \sum_{\alpha \neq 0} \frac{1}{e^{(\epsilon_\alpha - \epsilon_0)/(k_B T_{BEC})} - 1} \, , \tag{13}$$

while for $T < T_{BEC}$ we have

$$N = N_0 + \sum_{\alpha \neq 0} \frac{1}{e^{(\epsilon_\alpha - \epsilon_0)/(k_B T)} - 1} \, . \tag{14}$$

Then the condensate fraction reads

$$\frac{N_0}{N} = 1 - \frac{\sum_{\alpha \neq 0} \frac{1}{e^{(\epsilon_\alpha - \epsilon_0)/(k_B T)} - 1}}{\sum_{\alpha \neq 0} \frac{1}{e^{(\epsilon_\alpha - \epsilon_0)/(k_B T_{BEC})} - 1}} \, . \tag{15}$$

Statistical physics of trapped bosons (V)

Let us now consider a Bose gas of atoms with mass m in a hypercubic box of volume L^D. The single-particle energy spectrum is

$$\epsilon_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} \ . \tag{16}$$

In the continuum limit

$$\sum_{\mathbf{k}} \to L^D \int \frac{d^D\mathbf{k}}{(2\pi)^D} \tag{17}$$

and the total number density $n = \frac{N}{L^D}$ in the BEC phase is given by

$$n = n_0 + \int \frac{d^D\mathbf{k}}{(2\pi)^D} \frac{1}{e^{\frac{\hbar^2 k^2}{2mk_B T}} - 1} \ , \tag{18}$$

where $n_0 = N_0/L^D$ is the condensate number density.

Statistical physics of trapped bosons (VI)

Let us consider specifically the three-dimensional (3D) uniform bosonic gas of non-interacting spinless particles in a box.
We will calculate the critical temperature T_{BEC} of Bose-Einstein condensation as a function of the number density n of bosons and the condensate fraction n_0/n as a function of the temperature T.
In the three-dimensional case $(D = 3)$ from the equation

$$n = n_0 + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{e^{\frac{\hbar^2 k^2}{2mk_B T}} - 1} ,$$

we find

$$n = n_0 + \zeta(3/2) \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2}$$

with $\zeta(x)$ the Riemann zeta function and $\zeta(3/2) = 2.6124$.

Statistical physics of trapped bosons (VII)

It follows that

$$\frac{n_0}{n} = 1 - \frac{\zeta(3/2)\left(\frac{mk_B}{2\pi\hbar^2}\right)^{3/2} T^{3/2}}{n} = 1 - \frac{\zeta(3/2)\left(\frac{mk_B}{2\pi\hbar^2}\right)^{3/2} T^{3/2}}{\zeta(3/2)\left(\frac{mk_B}{2\pi\hbar^2}\right)^{3/2} T_{BEC}^{3/2}} .$$

Thus, the condensate fraction reads

$$\frac{n_0}{n} = 1 - \left(\frac{T}{T_{BEC}}\right)^{3/2} .$$

The critical temperature of Eq. (19) and this formula for the condensate fraction of non-interacting massive bosons were obtained in 1925 by Albert Einstein extending previous results derived by Satyendra Nath Bose for a gas a photons (massless bosons).

Statistical physics of trapped bosons (VIII)

More generally, for D spatial dimensions one finds for the the critical temperature T_{BEC} of Bose-Einstein condensation

$$k_B\, T_{BEC} = \begin{cases} \frac{2\pi}{\zeta(3/2)^{2/3}} \frac{\hbar^2}{m} n^{2/3} & \text{for } D = 3 \\ 0 & \text{for } D = 2 \\ \text{no solution} & \text{for } D = 1 \end{cases} \qquad (19)$$

where

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{y^{x-1}}{e^y - 1}\, dy \qquad (20)$$

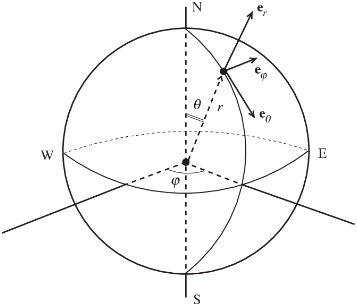
is the Riemann zeta function,

$$\Gamma(x) = \int_0^\infty y^{x-1}\, e^{-y}\, dy \qquad (21)$$

is the Euler gamma function, and $\zeta(3/2) = 2.6124$.

Bose gas on the surface of a sphere (I)

The theoretical study of a Bose gas on the surface of a sphere is triggered by the experimental confinement the atoms on a bubble trap,[6] which needs microgravity conditions.[7]



Spherical coordinates: radial coordinate $r \in [0, +\infty[$, polar angle $\theta \in [0, \pi]$, azimuthal angle $\phi \in [0, 2\pi]$. If the particle is constrained on the surface of the sphere, $r = R$, with R the radius of the sphere.

[6] B. M. Garraway and H. Perrin, J. Phys. B **49**, 172001 (2016).

[7] E.R. Elliott et al., npj Micrography **4**, 16 (2018); R.A. Carollo et al., R.A. Carollo et al., Nature **606**, 281 (2022).

Bose gas on the surface of a sphere (II)

Let us consider the single-particle Hamiltonian

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{\hbar^2 r^2}\right] \tag{22}$$

where $\hat{L}^2$ is the square of the orbital angular momentum operator. Under the assumption that the spherical radial coordinate r is fixed and given by

$$r = R\ , \tag{23}$$

the Hamiltonian becomes

$$\hat{H}_0 = \frac{\hat{L}^2}{2mR^2}\ . \tag{24}$$

Bose gas on the surface of a sphere (III)

Its eigenvalue problem reads

$$\hat{H}_0|\ell, m_\ell\rangle = \epsilon_\ell \,|\ell, m_\ell\rangle \tag{25}$$

because the eigenvalues of $\hat{L}^2$ are $\hbar^2\ell(\ell+1)$ with

$$\epsilon_\ell = \frac{\hbar^2}{2mR^2}\ell(\ell+1)\ . \tag{26}$$

Thus, the energy of a particle of mass m moving on the surface of a sphere of radius R is quantized according to this formula where $\ell = 0, 1, 2, ...$ is the **integer quantum number** of the angular momentum. This energy level has the degeneracy $2\ell + 1$ due to the magnetic quantum number $m_\ell = -\ell, -\ell+1, ..., \ell-1, \ell$ of the third component of the angular momentum.

Bose gas on the surface of a sphere (IV)

In quantum statistical mechanics the total number N of non-interacting bosons moving on the surface of a sphere and at equilibrium with a thermal bath of absolute temperature T is given by

$$N = \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{e^{(\epsilon_\ell - \mu)/(k_B T)} - 1} , \quad (27)$$

where k_B is the Boltzmann constant and μ is the chemical potential. In the Bose-condensed phase, we can set $\mu = 0$ and

$$N = N_0 + \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{e^{\epsilon_\ell/(k_B T)} - 1} , \quad (28)$$

where N_0 is the number of bosons in the lowest single-particle energy state, i.e. the number of bosons in the Bose-Einstein condensate (BEC).

Bose gas on the surface of a sphere (V)

Within the semiclassical approximation, where $\sum_{\ell=1}^{\infty} \to \int_1^{\infty} d\ell$, the previous equation becomes

$$n = n_0 + \frac{mk_B T}{2\pi\hbar^2} \left(\frac{\hbar^2}{mR^2 k_B T} - \ln \left(e^{\hbar^2/(mR^2 k_B T)} - 1 \right) \right) , \qquad (29)$$

where $n = N/(4\pi R^2)$ is the 2D number density and $n_0 = N_0/(4\pi R^2)$ is the 2D condensate density.
At the critical temperature T_{BEC}, where $n_0 = 0$, one then finds[8]

$$k_B T_{BEC} = \frac{\frac{2\pi\hbar^2}{m} n}{\frac{\hbar^2}{mR^2 k_B T_{BEC}} - \ln \left(e^{\hbar^2/(mR^2 k_B T_{BEC})} - 1 \right)} . \qquad (30)$$

As expected, in the limit $R \to +\infty$ one gets $T_{BEC} \to 0$, in agreement with the Mermin-Wagner theorem.[9] However, for any finite value of R the critical temperature T_{BEC} is larger than zero.

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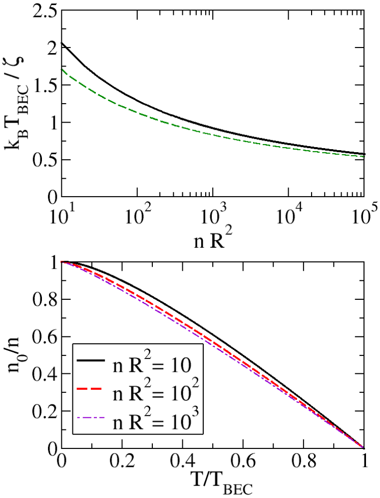
[8] A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).
[9] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

Bose gas on the surface of a sphere (VI)



Top panel: T_{BEC} vs nR^2, with $\zeta = \hbar^2 n/m$. Solid line: semiclassical approximation (solid line); dashed line: numerical evaluation of the sum. **Bottom panel**: condensate fraction n_0/n vs temperature T/T_{BEC}.

Bose gas on the surface of a sphere (VII)

We now consider a system of interacting bosons on the surface of a sphere of radius R and contact interaction of strength g[10], i.e.

$$V(\mathbf{r}-\mathbf{r}') \simeq g\,\delta^{(2)}(\mathbf{r}-\mathbf{r}')\;. \tag{31}$$

Within a perturbative scheme[11] generalizing the previous equations we obtain the BEC critical temperature

$$k_B T_{BEC} = \frac{\frac{2\pi\hbar^2 n}{m} - \frac{g n}{2}}{\frac{\hbar^2}{2mR^2 k_B T_{BEC}}\left(1+\sqrt{1+\frac{2g mnR^2}{\hbar^2}}\right) - \ln\left(e^{\frac{\hbar^2}{mR^2 k_B T_{BEC}}\sqrt{1+\frac{2g mnR^2}{\hbar^2}}} - 1\right)}\,, \tag{32}$$

where the condensate density n_0 is zero.

[10] A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).
[11] H. Kleinert, S. Schmidt, and A. Pelster, Phys. Rev. Lett. **93**, 160402 (2004).

Statistical physics of trapped fermions (I)

In quantum statistical mechanics the total number N of trapped non-interacting fermions (with two spin components) moving on the surface of a sphere and at equilibrium with a thermal bath of absolute temperature T is given by

$$N = 2 \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{e^{(\epsilon_\ell - \mu_\parallel)/(k_B T)} + 1} \,, \tag{33}$$

where k_B is the Boltzmann constant and $\mu_\parallel$ is the chemical potential of the 2D system.

We now use the Euler-MacLaurin formula

$$\sum_{\ell=0}^{\infty} f(l) = \int_0^{+\infty} dl\, f(l) + \frac{1}{2} f(0) - \frac{1}{12} \frac{df}{dl}(0) + \dots \tag{34}$$

In this way we get the number density

$$n = \frac{N}{4\pi R^2} = \frac{m k_B T}{\pi \hbar^2} \ln\left(1 + e^{\mu_\parallel/(k_B T)}\right) + \frac{1}{6\pi R^2} \frac{e^{\mu_\parallel/(k_B T)}}{\left(1 + e^{\mu_\parallel/(k_B T)}\right)} + \dots \tag{35}$$

that is the familiar result of the 2D flat space plus finite-size corrections which depend on the radius R. As expected, in the thermodynamic limit $R \to +\infty$ only the flat term survives.

Statistical physics of trapped fermions (II)

Notice that in the zero-temperature limit $T \to 0^+$ we have

$$n = \frac{m}{\pi\hbar^2}\mu_{\parallel} + \frac{1}{6\pi R^2} + \ldots \, , \tag{36}$$

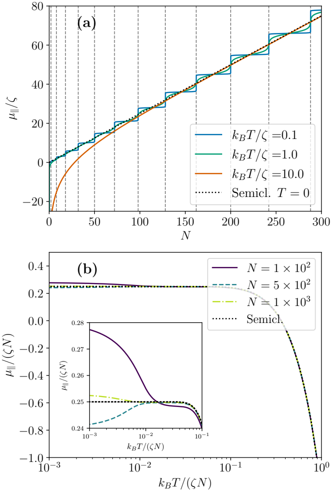
namely (with $nR^2 \gg 1/(6\pi)$)

$$\mu_{\parallel} = \frac{\pi\hbar^2 n}{m} - \frac{\hbar^2}{6mR^2} + \ldots \, . \tag{37}$$

This is the Fermi energy of a two-component ideal Fermi gas with 2D number density n moving a the superface of a sphere of radius R. One can then derive many other thermodynamical quantities. For instance, the internal energy density at zero temperature reads

$$\frac{E}{4\pi R^2} = \int_0^n \mu_{\parallel}(\tilde{n})\, d\tilde{n} = \frac{\pi\hbar^2 n^2}{2m} - \frac{\hbar^2 n}{6mR^2} + \ldots \, . \tag{38}$$

Statistical physics of fermions (III)



Energy scale $\zeta = \hbar^2/(mR^2)$. Figure adapted from L. Frigato, A. Bardin, and LS, arXiv:2512.16384, accepted in Phys. Rev. A (2026).

Fermi gas on the surface of a sphere (I)

Currently we are studying the problem of two-spin-component interacting fermions moving on the surface of a sphere of radius R, both in the repulsive[12] and attractive case.[13]

For fermions the grand canonical partition function $\mathcal{Z}$ can be written in the framework of functional integration as follows

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}_\sigma, \psi_\sigma] \, e^{-\frac{S_E[\bar{\psi}_\sigma, \psi_\sigma]}{\hbar}} \tag{39}$$

where the functional integration involves the Berezin integral of Grassmann fields, which appear in the Euclidean action S_E.

The grand potential is then given by

$$\Omega = -k_B T \, \ln(\mathcal{Z}) \, , \tag{40}$$

while the average number of fermions reads

$$N = -\left(\frac{\partial \Omega}{\partial \mu_\parallel}\right)_{T,R} \, . \tag{41}$$

[12] L. Frigato, A. Bardin, and LS, arXiv:2512.16384, accepted in Phys. Rev. A (2026).

[13] L. Frigato, A. Tononi, and LS, in preparation.

Fermi gas on the surface of a sphere (II)

In the grand canonical ensemble the Euclidean action of fermions with contact interaction on the surface of a sphere is given by

$$S_E[\bar{\psi}_\sigma, \psi_\sigma] = \int_0^{\hbar/(k_B T)} d\tau \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin(\theta) \, \mathscr{L}_E(\bar{\psi}_\sigma, \psi_\sigma) \quad (42)$$

where the Euclidean Lagrangian density reads

$$\mathscr{L}_E = \sum_{\sigma=\uparrow,\downarrow} \bar{\psi}_\sigma \left(\hbar \frac{\partial}{\partial \tau} + \frac{\hat{L}^2}{2mR^2} - \mu_{\|} \right) \psi_\sigma + g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \quad (43)$$

with $\psi_\sigma(\theta, \phi, \tau)$ the Grassman field, which depends on the angular variables θ and ϕ, and also on the imaginary time τ. $\mu_{\|}$ is the chemical potential and g is the interaction strength.

Fermi gas on the surface of a sphere (IV)

Repulsive fermions can be investigated by using the Hartree-Fock approximation:

$$\bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \simeq \frac{\tilde{n}}{2} \bar{\psi}_\uparrow \psi_\uparrow + \frac{\tilde{n}}{2} \bar{\psi}_\downarrow \psi_\downarrow - \frac{\tilde{n}^2}{4} \ , \tag{44}$$

assuming a balanced configuration

$$\frac{\tilde{n}}{2} = \tilde{n}_\uparrow = \tilde{n}_\downarrow \tag{45}$$

with

$$\tilde{n}_\sigma = \langle \bar{\psi}_\sigma \psi_\sigma \rangle \ . \tag{46}$$

In this way the Hartre-Fock Euclidean Lagrangian density is quadratic with respect to the fermionic fields

$$\mathscr{L}_{E,\mathrm{HF}} = \sum_{\sigma=\uparrow,\downarrow} \bar{\psi}_\sigma \left(\hbar \frac{\partial}{\partial \tau} + \frac{\hat{L}^2}{2mR^2} - \mu_\parallel + g \frac{\tilde{n}}{2} \right) \psi_\sigma - g \frac{\tilde{n}^2}{4} \tag{47}$$

and the corresponding Gaussian functional integrals can be exactly calculated. Notice that $n = \tilde{n}/R^2$.

Conclusions

At the end of this lecture I hope you may comment using the words of Enrico Fermi:

"Before I came here I was confused about this subject.
Having listened to your lectures I am still confused.
But on a higher level."

THANK YOU VERY MUCH FOR YOUR ATTENTION!