Unitary Fermi Superfluid: Thermodynamics and Sound Modes from Elementary Excitations

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Summary

- Unitary Fermi gas
- Single-particle and collective excitations
- Universal thermodynamics
- Superfluid fraction and critical temperature
- First and second sound
- Sound mixing
- Conclusions

Unitary Fermi gas (I)

Let us consider a **gas of atomic fermions** with two equally-populated spin components: $n_\uparrow = n_\downarrow$. The system is dilute if the effective range r_e of the inter-atomic potential is much smaller than the average interparticle separation $d = n^{-1/3}$ with total number density $n = n_\uparrow + n_\downarrow$, namely

$$r_e \ll d \ . \tag{1}$$

The system is strongly-interacting if the scattering length a of the inter-atomic potential greatly exceeds the average interparticle separation $d = n^{-1/3}$, i.e.

$$d \ll |a| \ . \tag{2}$$

The unitarity regime[1] is characterized by both these conditions:

$$r_e \ll d \ll |a| \ . \tag{3}$$

Under these conditions the dilute but strongly-interacting Fermi gas is called **unitary Fermi gas**.

[1] S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 80, 1215 (2008).

Unitary Fermi gas (II)

Ideally, the unitarity limit corresponds to

$$r_e = 0 \qquad \text{and} \qquad a = \pm\infty \, . \tag{4}$$

In a uniform configuration and at zero temperature, the only length characterizing the Fermi gas in the unitarity limit is the average interparticle distance $d = n^{-1/3}$.
In this case the ground-state energy must be[2]

$$E_0 = \xi \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n^{2/3} N = \xi \frac{3}{5} \epsilon_F N \tag{5}$$

with $\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2)^{2/3} n^{2/3}$ Fermi energy of the ideal gas and ξ a universal unknown parameter (Bertsch parameter).
Monte Carlo calculations and experimental data with dilute and ultracold atoms suggest that, at zero temperature, the unitary Fermi gas is a superfuid with $\xi \simeq 0.4$.

[2] W. Zwerger (Ed.), The BCS-BEC Crossover and the Unitary Fermi Gas (Springer, 2011).

Single-particle and collective excitations (I)

Inspired by the Landau theory of elementary excitations we model the many-body quantum Hamiltonian $\hat{H}$ of the uniform unitary Fermi gas with the simple effective Hamiltonian

$$\hat{H} = E_0 + \sum_{\sigma=\uparrow,\downarrow} \sum_{\mathbf{k}} \epsilon_{sp}(k)\ \hat{c}^\dagger_{\mathbf{k}\sigma} \hat{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} \epsilon_{col}(q)\ \hat{b}^\dagger_{\mathbf{q}} \hat{b}_{\mathbf{q}} \ , \qquad (6)$$

where
the $\hat{c}^\dagger_{\mathbf{k}\sigma}$ ($\hat{c}_{\mathbf{k}\sigma}$) operator creates (annihilates) a single-particle excitation, respectively, with linear momentum $\mathbf{k}$, spin σ, and energy $\epsilon_{sp}(k)$,
whereas
the $\hat{b}^\dagger_{\mathbf{q}}$ ($\hat{b}_{\mathbf{q}}$) operator creates (annihilates) a bosonic collective excitation, respectively, of linear momentum $\mathbf{q}$ and energy $\epsilon_{col}(q)$.

Single-particle and collective excitations (II)

The dispersion of the BCS-like single-particle elementary excitations can be written as

$$\epsilon_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \zeta\epsilon_F\right)^2 + \Delta_0^2} \qquad (7)$$

where ζ is a parameter taking into account the interaction between fermions and the reconstruction of the Fermi surface close to the critical temperature. In particular, $\zeta = 0.9$ according[3] to accurate Monte Carlo results. Moreover, Δ_0 is the gap parameter, with $2\Delta_0$ the minimal energy to break a Cooper pair. The gap energy Δ_0 of the unitary Fermi gas at zero-temperature has been calculated with Monte Carlo simulations[4] and found to be

$$\Delta_0 = \gamma\epsilon_F \qquad (8)$$

with $\gamma = 0.45$.

[3] P. Magierski, G. Wlazlowski, A. Bulgac, and J. E. Drut, Phys. Rev. Lett. **103**, 210403 (2009).

[4] J. Carlson and S. Reddy, Phys. Rev. Lett. **95**, 060401 (2005).

Single-particle and collective excitations (III)

The dispersion relation of collective elementary excitations is assumed[5] to be given by

$$\epsilon_{\rm col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(2mc_B^2 + \lambda \frac{\hbar^2 q^2}{2m}\right)} \, , \qquad (9)$$

where $c_B = \sqrt{\xi/3}\ v_F$ is the Bogoliubov sound velocity with $v_F = \sqrt{2\epsilon_F/m}$ the Fermi velocity of a non-interacting Fermi gas.
In a old paper [LS, Phys. Rev. A **82**, 063619 (2010)] we used the value $\lambda = 0.25$, which is consistent with a macroscopic time-dependent nonlinear Schrödinger equation approach without the inclusion of spurious terms.[6]
In a recent paper [G. Bighin, A. Cappellaro, and LS, Phys. Rev. A **105**, 063329 (2022)] we have used instead $\lambda = 0.08$, which is the value obtained[7] from the beyond-mean-field GPF theory[8] at unitarity.

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[5] LS, Phys. Rev. A **82**, 063619 (2010).
[6] LS and F. Toigo, Phys. Rev. A 78, 053626 (2010).
[7] G. Bighin, LS, P. A. Marchetti, and F. Toigo, Phys. Rev. A 92, 023638 (2015).
[8] J. Tempere and J. P. Devreese, Superconductors: Materials, Properties and Applications, InTech 383 (2012).

Universal thermodynamics (I)

The Helmholtz free energy F of the system is given by the usual formula $F = -k_B T \ln \mathcal{Z}$, where we introduced the partition function $\mathcal{Z}$ of the system, defined as

$$\mathcal{Z} = \mathrm{Tr}[e^{-\hat{H}/k_B T}] \, . \tag{10}$$

Similarly to Eq. (6), the free energy of the unitary Fermi gas can be written as $F = F_0 + F_{sp} + F_{col}$, where F_0 is the free energy of the ground-state,

$$F_{sp} = -2 k_B T \sum_{\mathbf{k}} \ln[1 + e^{-\epsilon_{sp}(k)/(k_B T)}] \tag{11}$$

is the free energy of fermionic single-particle excitations and finally

$$F_{col} = -k_B T \sum_{\mathbf{q}} \ln[1 - e^{-\epsilon_{col}(q)/(k_B T)}] \tag{12}$$

is the free energy of the bosonic collective excitations.

Universal thermodynamics (II)

The total Helmholtz free energy F of a unitary Fermi gas in the superfluid phase can be then written[9] as

$$F = N\epsilon_F \Phi(x) \, , \tag{13}$$

where, due to the scale-invariance of the system, $\Phi(x)$ is a function of the scaled temperature $x \equiv T/T_F$ only, having defined the Fermi temperature $T_F = \epsilon_F / k_B$. Explicitly, $\Phi(x)$ takes the following form

$$\begin{aligned} \Phi(x) &= \frac{3}{5}\xi - 3x \int_0^{+\infty} \ln\left[1 + e^{-\tilde{\epsilon}_{\rm sp}(u)/x}\right] u^2 \mathrm{d}u \\ &+ \frac{3}{2}x \int_0^{+\infty} \ln\left[1 - e^{-\tilde{\epsilon}_{\rm col}(u)/x}\right] u^2 \mathrm{d}u \, . \end{aligned} \tag{14}$$

Note that the discrete summations have been replaced by integrals, and that we set $\tilde{\epsilon}_{\rm col}(u) = \sqrt{u^2(4\xi/3 + \lambda u^2)}$ and $\tilde{\epsilon}_{\rm sp}(u) = \sqrt{(u^2 - \zeta)^2 + \gamma^2}$.

[9] LS, Phys. Rev. A **82**, 063619 (2010).

Universal thermodynamics (III)

We now aim at calculating the thermodynamics of the system in terms of the universal function $\Phi(x)$ and its derivatives. From the Helmholtz free energy F we can immediately obtain the chemical potential μ, that is defined as

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = \epsilon_F \left[\frac{5}{3}\Phi\left(\frac{T}{T_F}\right) - \frac{2}{3}\frac{T}{T_F}\Phi'\left(\frac{T}{T_F}\right)\right], \qquad (15)$$

where $\Phi'(x) = \frac{d\Phi(x)}{dx}$ and one recovers $\mu_0 = \xi\epsilon_F$ in the limit of zero-temperature.

The entropy S is readily calculated from the free energy F through the relation

$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V} = -Nk_B\Phi'(x)\,. \qquad (16)$$

where $\Phi'(x)$ is the first derivative of Φ with respect to x.

Universal thermodynamics (IV)

Furthermore, the internal energy $E = F + TS$, can immediately be rewritten as

$$E = N\epsilon_F \left[\Phi(x) - x\ \Phi'(x)\right] \tag{17}$$

and, similarly, the pressure P is related to the free energy F by the simple relation

$$P = -\left(\frac{\partial F}{\partial V}\right)_{N,T} = \frac{2}{3} n\epsilon_F \left[\Phi(x) - x\Phi'(x)\right] \ . \tag{18}$$

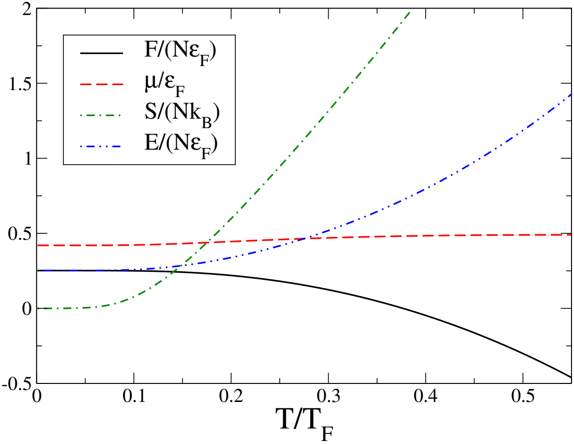
Remark: Adopting the Maxwell-Boltzmann distribution for fermionic single-particles instead of the Fermi-Dirac one, and under the further assumption that $\lambda = 0$, the adimensional fee energy becomes

$$\Phi(x) \simeq \frac{3}{5}\xi - \frac{\pi^4\sqrt{3}}{80\ \xi^{3/2}} x^4 - \frac{3\sqrt{2\pi}}{2} \zeta^{1/2} \gamma^{1/2} x^{3/2} e^{-\gamma/x} \ . \tag{19}$$

This expression was proposed by Bulgac, Drut and Magierski.[10] We call this equation the BDM model.

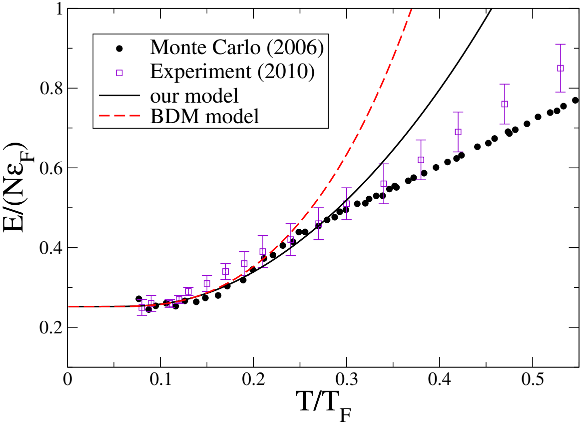
[10] A. Bulgac, J.E. Drut, and P. Magierski, Phys. Rev. Lett **96**, 090404 (2006).

Universal thermodynamics (V)



Thermodynamical quantities of the unitary Fermi gas deduced from our model as a function of the adimensional temperature T/T_F with $T_F = \epsilon_F/k_B$ the Fermi temperature. Plot taken from LS, Phys. Rev. A **82**, 063619 (2010), where $\xi = 0.42$, $\lambda = 0.25$, $\zeta = 0.9$, and $\gamma = 0.45$.

Universal thermodynamics (VI)



Scaled internal energy $E/(N\epsilon_F)$ as a function of the scaled temperature T/T_F. Filled circles: Monte Carlo simulations [Phys. Rev. A **78**, 023625 (2008)]. Open squares with error bars: experimental data [Science **442**, 327 (2010)]. Solid line: our model with $\xi = 0.42$, $\lambda = 0.25$, $\zeta = 0.9$, and $\gamma = 0.45$. Dashed line: Bulgac-Drut-Magierski (BDM) model. Plot taken from LS, Phys. Rev. A **82**, 063619 (2010).

Superfluid fraction and critical temperature (I)

According to Landau's two fluid theory[11] the total number density n of a system in the superfluid phase can be written as

$$n = n_{\rm s} + n_{\rm n} \ , \tag{20}$$

where $n_{\rm s}$ is the superfluid density and $n_{\rm n}$ is the normal density. Naturally, at zero temperature the whole system is in the superfluid phase, and one has $n_{\rm n} = 0$ and $n = n_{\rm s}$. As the temperatures increases, the normal density $n_{\rm n}$ increases, as well, until at the critical temperature T_c one has $n_{\rm n} = n$ and, correspondingly, $n_{\rm s} = 0$. Within our scheme, the normal density of a unitary gas is given the sum of two contributions

$$n_{\rm n} = n_{\rm n,sp} + n_{\rm n,sp} \ , \tag{21}$$

i.e. a contribution $n_{\rm n,sp}$ from to the single-particle excitations and a contribution $n_{\rm n,col}$ from collective excitations.

[11] L.D. Landau, J. Phys. (USSR) **5**, 71 (1941).

Superfluid fraction and critical temperature (II)

Landau linked the normal densities to their statistic and their energy spectrum, so that in the present case the single-particle contribution to the normal density reads

$$n_{\mathrm{n,sp}} = \frac{1}{k_B T V} \sum_{\mathbf{k}} k^2 \frac{e^{\epsilon_{\mathrm{sp}}(k)/(k_B T)}}{(e^{\epsilon_{\mathrm{sp}}(k)/(k_B T)} + 1)^2} \ , \tag{22}$$

whereas, concerning the contribution from the collective modes,

$$n_{\mathrm{n,col}} = \frac{1}{2 k_B T V} \sum_{\mathbf{q}} q^2 \frac{e^{\epsilon_{\mathrm{col}}(q)/(k_B T)}}{(e^{\epsilon_{\mathrm{col}}(q)/(k_B T)} - 1)^2} \ . \tag{23}$$

Superfluid fraction and critical temperature (III)

It is then easy to derive the superfluid fraction

$$\frac{n_s}{n} = 1 - \Xi(x) \ , \tag{24}$$

where the universal function $\Xi(x)$ is again a function of the scaled temperature $x \equiv T/T_F$ only, explicitly given by

$$\begin{aligned} \Xi(x) &= \frac{2}{x}\int_0^{+\infty} \frac{e^{\tilde{\epsilon}_{sp}(\eta)/x}}{(e^{\tilde{\epsilon}_{sp}(\eta)/x}+1)^2}\ \eta^4 \mathrm{d}\eta \\ &+ \frac{1}{x}\int_0^{+\infty} \frac{e^{\tilde{\omega}_{col}(\eta)/x}}{(e^{\tilde{\omega}_{col}(\eta)/x}-1)^2}\ \eta^4 \mathrm{d}\eta \ , \end{aligned} \tag{25}$$

where we have converted sums to integrals.

Superfluid fraction and critical temperature (IV)

The superfluid density defines the critical temperature T_c of the superfluid-to-normal phase transition via the condition $n_s = 0$. With our choice of parameters for the elementary excitations we find

$$T_c = 0.23 \; T_F \; . \tag{26}$$

It must be pointed out that, while this estimation of the critical temperature agrees with more refined approaches, such as the functional GPF theory[12] or the NSR scheme,[13] it actually differs from the most recent experimental results,[14] placing it at $T_c/T_F \simeq 0.17$.
The overestimation of our theoretical critical temperature with respect to the experimental ones does not appear plotting the physical quantities vs T/T_c.

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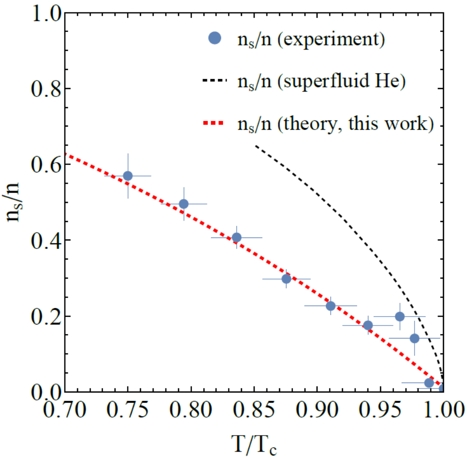
[12] H. Hu, X. J. Liu, and P. D. Drummond, EPL **74**, 574 (2007); J. Tempere and J. P. Devreese, Superconductors: Materials, Properties and Applications, InTech 383 (2012).

[13] P. Nozieres and S. Schmitt-Rink, J. Low. Temp. Phys. 59, 195 (1985).

[14] X. Li *et al.*, Science **375**, 528 (2022).

Superfluid fraction and critical temperature (V)



Superfluid fraction n_s/n as a function of the adimensional temperature T/T_c. Comparison between our theory and recent experimental data [X. Li *et al.*, Science **375**, 528 (2022)]. Plot adapted from G. Bighin, A. Cappellaro, and LS, Phys. Rev. A **105**, 063329 (2022), where $\xi = 0.38$, $\lambda = 0.08$, $\zeta = 0.9$, and $\gamma = 0.45$.

First and second sound (I)

According to Landau, in a superfluid a local perturbation excites two wave-like modes the first and the second sound which propagate with velocities u_1 and u_2. These velocities are determined by the positive solutions of the algebraic biquadratic equation

$$u^4 + (c_{10}^2 + c_{20}^2) u^2 + c_T^2 c_{20}^2 = 0 \, , \tag{27}$$

where

$$c_{10} = \sqrt{\frac{1}{m} \left(\frac{\partial P}{\partial n} \right)_{\bar{S},V}} = v_F \sqrt{\frac{5}{9} \Phi(x) - \frac{5}{9} \frac{T}{T_F} \Phi'(x)} \tag{28}$$

is the adiabatic sound velocity with $\bar{S} = S/N$ the entropy per particle,

$$c_{20} = \sqrt{\frac{1}{m} \frac{\bar{S}^2}{\left(\frac{\partial \bar{S}}{\partial T} \right)_{N,V}} \frac{n_s}{n_n}} = v_F \sqrt{-\frac{1}{2} \frac{\Phi'(x)^2}{\Phi''(x)} \frac{1 - \Xi(x)}{\Xi(x)}} \tag{29}$$

is the entropic sound velocity, and

$$c_T = \sqrt{\frac{1}{m} \left(\frac{\partial P}{\partial n} \right)_{T,V}} = v_F \sqrt{\frac{5}{9} \Big(\Phi(x) - \frac{T}{T_F} \Phi'(x) \Big) + \frac{2}{9} x^2 \Phi''(x)} \tag{30}$$

is the isothermal sound velocity.

First and second sound (II)

The first sound u_1 is the largest of the two positive roots of Eq. (27) while the second sound u_2 is the smallest positive one. Thus

$$u_{1,2} = \sqrt{\frac{c_{10}^2 + c_{20}^2}{2} \pm \sqrt{\left(\frac{c_{10}^2 + c_{20}^2}{2}\right)^2 - c_{20}^2 c_T^2}} \, . \qquad (31)$$

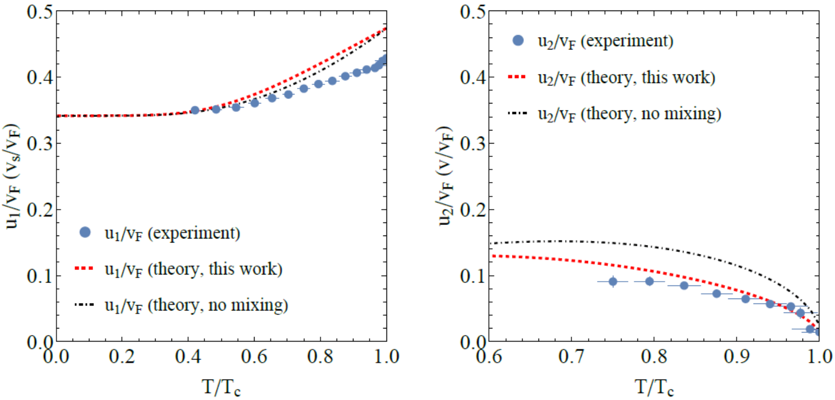
For the sake of completeness, we stress that the "Einstein-like relation"

$$\frac{E}{N} = \frac{10}{9} m c_{10}^2 \qquad (32)$$

derived in a recent paper[15] is automatically verified within our universal thermodynamic formalism, that naturally includes the scale-invariance of the unitary Fermi gas.

[15] P. B. Patel *et al.*, Science **370**, 1222 (2020).

First and second sound (III)



First sound velocity u_1 and second sound velocity u_2 as a function of the adimensional temperature T/T_c. Here $v_F = \sqrt{2\epsilon_F/m}$ is the Fermi velocity. Comparison between our theory and recent experimental data [X. Li *et al.*, Science **375**, 528 (2022)]. "No mixing" means the (wrong) assumption that $c_T = c_{10}$. Plot adapted from G. Bighin, A. Cappellaro, and LS, Phys. Rev. A **105**, 063329 (2022).

Sound mixing (I)

It is useful to analyze the amplitudes modes $W_1$and W_2 of the response to a density perturbation,[16] i.e.

$$\delta n(x,t) = W_1 \delta n_1(x \pm u_1 t) + W_2 \delta n_2(x \pm u_2 t) \tag{33}$$

where

$$\frac{W_1}{W_1 + W_2} = \frac{(u_1^2 - c_{20}^2)\, u_2^2}{(u_1^2 - u_2^2)\, c_{20}^2} \tag{34}$$

and

$$\frac{W_2}{W_1 + W_2} = \frac{(c_{20}^2 - u_2^2)\, u_1^2}{(u_1^2 - u_2^2)\, c_{20}^2} \ . \tag{35}$$

[16] T. Ozawa and S. Stringari, Phys. Rev. Lett. **112**, 025302 (2014).

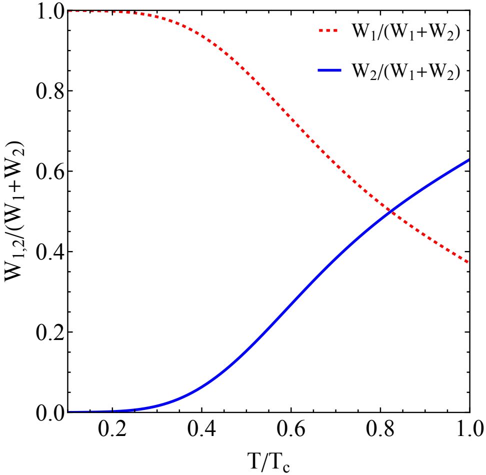
Sound mixing (II)

Superfluid ^{4}He is characterized by "no mode mixing" (i.e. $c_T \simeq c_{10}$): the first sound corresponds to a standard density waves (in-phase oscillations of the superfluid and normal components) and the second sound is understood as an entropy wave. It is important to notice that, under the "no-mixing condition" $c_{10} = c_T$, Eqs. (34) and (35) read $W_1 = 1$ and $W_2 = 0$. This means that, in this case, a density probe excites only the first sound mode.

For the unitary Fermi gas the situation is radically different[17] because the isothermal velocity c_T and the adiabatic velocity c_{10} are quite different. Thus, we expect that $W_1 \neq 1$ and $W_2 \neq 0$ for the unitary Fermi gas.

[17] L. P. Pitaevskii and S. Stringari, pp. 322-347, in Universal Themes of Bose-Einstein Condensation Edited by N.P. Proukakis, D.W. Snoke, and P.B. Littlewood (Cambridge University Press, 2017).

Sound mixing (III)



Contribution from the first (dashed red line) and second sound (**solid blue line**) to the amplitude of a density response as a function of the scaled temperature T/T_c. Figure adapted from G. Bighin, A. Cappellaro, and LS, Phys. Rev. A **105**, 063329 (2022).

Conclusions

- A simple description in terms of fermionic single-particle and bosonic collective elementary excitations is able to reproduce many properties of the **unitary Fermi gas**.
- The internal energy derived from our model is in good agreement with Monte Carlo simulations and experimental results for $T \leq 0.25 T_F$.
- We have reproduced the recently-measured superfluid fraction, first sound and second sound near the critical temperature $T_c \simeq 0.2 T_F$.
- Contrary to Helium 4, near the critical temperature the first and second sound of the **unitary Fermi gas** cannot be interpreted as a pure pressure-density wave and a pure entropy-temperature wave, respectively.
- Our investigation of the **unitary Fermi gas** shows that at very low temperatures the mixing of pressure-density and entropy-temperature oscillations is absent, whereas approaching T_c a density probe will excite both sounds.

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