Berezinskii-Kosterlitz-Thouless Phase Transition with Rabi Coupled Bosons

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LPHYS’23, July 3, 2023

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Summary

- 2D Bose-Bose mixture with Rabi coupling
- Elementary excitations
- Superfluid density
- Renormalized superfluid density
- BKT critical temperature
- First and second sound
- Conclusions

2D Bose-Bose mixture with Rabi coupling (I)

We consider a 2D atomic Bose gas confined in a quadratic region of area L^2. The bosonic gas is characterized by two-hyperfine components with bosonic complex fields $\psi_a(\mathbf{r}, t)$, $a = 1, 2$. The Lagrangian density of the system reads

$$\begin{aligned} \mathcal{L} &= \sum_{a=1,2} \left[i\hbar\psi_a^* \partial_t \psi_a - \frac{\hbar^2}{2m} |\nabla\psi_a|^2 - \frac{g}{2} |\psi_a|^4 \right] - g_{12} |\psi_1|^2 |\psi_2|^2 \\ &+ \hbar\omega_R \left[\psi_1^* \psi_2 + \psi_2^* \psi_1 \right] \ . \end{aligned} \tag{1}$$

In addition to the usual intra-species ($g = g_{11} = g_{22}$) and inter-species (g_{12}) contact interactions, atoms with mass m in different hyperfine states interact also via an external coherent Rabi coupling of frequency ω_{R}, which drives an exchange of atoms between the two components.

2D Bose-Bose mixture with Rabi coupling (II)

The presence of the Rabi coupling implies that only the total number

$$N = N_1(t) + N_2(t) \tag{2}$$

of atoms is conserved, with

$$N_a(t) = \int_{L^2} |\psi_a(\mathbf{r}, t)|^2 \, d^2\mathbf{r} \tag{3}$$

the number of atoms in the a-th hyperfine component ($a = 1, 2$).
The existence and stability of the symmetric ground state with $N_1 = N_2$ have been previously discussed[1]. In addition to the symmetric configuration, a ground state with non-zero population imbalance is also possible[2]
Here we focus on the **symmetric and uniform configuration** analyzing finite temperature effects.

[1] M. Abad and A. Recati, Eur. Phys. J. D **67** (7), 148 (2013).
[2] C. P. Search, A. G. Rojo, and P. R. Berman, Phys. Rev. A **64**, 013615 (2001).

2D Bose-Bose mixture with Rabi coupling (III)

At the mean-field level, for the **symmetric and uniform ground state**, characterized by $n_1 = n_2 = n/2$, the chemical potential μ reads

$$\mu = \frac{1}{2} g n (1+\eta) - \hbar\omega_{\mathrm{R}} \ , \tag{4}$$

where $n = N/L^2$ is the 2D total number density of bosons, with $g > 0$ and $\eta = g_{12}/g$.
This **symmetric and uniform ground state** is stable under the conditions[3]

$$g + g_{12} > 0 \qquad \text{and} \qquad (g - g_{12}) n + 2\hbar\omega_{\mathrm{R}} > 0 \ , \tag{5}$$

namely

$$-1 < \eta < 1 + \frac{2\hbar\omega_{\mathrm{R}}}{gn} \ . \tag{6}$$

[3] M. Abad and A. Recati, Eur. Phys. J. D **67** (7), 148 (2013).

Elementary excitations (I)

At zero temperature, the Bogoliubov spectrum of elementary excitations of the uniform system has two branches[4], given by

$$E_k^{(-)} = \sqrt{\frac{\hbar^2 k^2}{2m}\left[\frac{\hbar^2 k^2}{2m} + 2(\mu + \hbar\omega_{\mathrm{R}})\right]}\,, \quad (7)$$

$$E_k^{(+)} = \sqrt{\frac{\hbar^2 k^2}{2m}\left[\frac{\hbar^2 k^2}{2m} + 2A\right] + B}\,, \quad (8)$$

where μ is the chemical potential. Moreover the two parameters appearing in the gapped branch are

$$A = \frac{1}{2} gn(1-\eta) + 2\hbar\omega_{\mathrm{R}}\,, \quad (9)$$

$$B = 4\hbar\omega_{\mathrm{R}}\left[\frac{1}{2} gn(1-\eta) + \hbar\omega_{\mathrm{R}}\right]\,, \quad (10)$$

again with $\eta = g_{12}/g$ and $g = g_{11} = g_{22}$.

[4] A. Cappellaro, T. Macrì, G. F. Bertacco, and LS, Sci. Rep. **7**, 13358 (2017).

Elementary excitations (II)

For small wavenumbers, i.e. for $k \ll 1$, the elementary excitations (7) and (8) read

$$E_k^{(-)} = \sqrt{\frac{gn(1+\eta)}{2m}}\ \hbar k \tag{11}$$

$$E_k^{(+)} = \sqrt{B} + \frac{A}{2\sqrt{B}} \frac{\hbar^2 k^2}{m} \ , \tag{12}$$

showing explicitly that the mode $E_k^{(-)}$ is gapless while the mode $E_k^{(+)}$ is gapped (if $\omega_{\mathrm{R}} \neq 0$). Notice that

$$c_{\mathrm{B}} = \sqrt{\frac{gn(1+\eta)}{2m}} \tag{13}$$

is the Bogoliubov speed of sound for the uniform system. For $\eta = 1$ one recovers the familiar expression $c_{\mathrm{B}} = \sqrt{gn/m}$, while $A = \sqrt{B} = 2\hbar\omega_{\mathrm{R}}$.

Superfluid density (I)

Adopting the Landau's approach[5], at finite temperature T the superfluid density $n_{\rm s}$ of the system can be written as

$$n_{\rm s}(T) = n - n_{\rm n}^{(-)}(T) - n_{\rm n}^{(+)}(T) \quad (14)$$

where

$$n_{\rm n}^{(-)}(T) = -\frac{1}{4}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{\hbar^2 k^2}{m} f'_T(E_k^{(-)}) \quad (15)$$

$$n_{\rm n}^{(+)}(T) = -\frac{1}{4}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{\hbar^2 k^2}{m} f'_T(E_k^{(+)}) \quad (16)$$

are the thermally activated normal densities due to the elementary excitations. In these formulas, $f'_T(E)$ is the derivative with respect to E of the Bose function $f_T(E) = 1/[e^{E/(k_{\rm B}T)} - 1]$ with $k_{\rm B}$ the Boltzmann constant and T the absolute temperature.

[5] L.D. Landau, J. Phys. (USSR) **5**, 71 (1941).

Superfluid density (II)

By using the low-momenta results (11) and (12), after integration we find

$$n_{\rm n}^{(-)}(T) = \frac{3\zeta(3)}{4\pi\hbar^2 m c_{\rm B}^4}(k_{\rm B}T)^3 \,, \qquad (17)$$

$$n_{\rm n}^{(+)}(T) = -\frac{m}{4\pi\hbar^2}\frac{B}{A^2}k_{\rm B}T \ln\left(1 - e^{-\frac{\sqrt{B}}{k_{\rm B}T}}\right) \,, \qquad (18)$$

where $\zeta(x)$ is the Riemann zeta function and $\zeta(3) = 1.20206$. For $\eta = 1$ the formula of the superfluid density becomes quite simple

$$n_{\rm s}(T) = n - \frac{3\zeta(3)m(k_{\rm B}T)^3}{4\pi\hbar^2(gn)^2} + \frac{mk_{\rm B}T}{4\pi\hbar^2}\ln\left(1 - e^{-\frac{2\hbar\omega_{\rm R}}{k_{\rm B}T}}\right) \,, \qquad (19)$$

because the role of the interaction is only encoded in the gapless excitation while the role of the Rabi coupling is only encoded in the gapped excitation.

Renormalized superfluid density (I)

It is important to stress that the superfluid density derived with the Landau formula does not take into account the formation of **quantized vortices**. This superfluid density goes to zero at a critical temperature that is larger than $T_{\rm c}$, the critical temperature of the Berezinskii-Kosterlitz-Thouless phase transition induced by the unbinding of vortex-antivortex pairs and the proliferation of free quantized vortices.[6]

Extending the approach of Kosterlitz and Thouless, we derive and solve Renormalization Group (RG) equations for our binary Bose mixture with balanced densities.[7] From these equations, which have the "bare" Landau superfluid density an an input, we obtained a renormalized superfluid density.

[6] V.L. Berezinskii, *Sov. Phys. JETP* **34**, 610 (1972); J. M. Kosterlitz and D. J. Thouless, J. Phys. C: Solid State Phys. **6**, 1181 (1973).

[7] K. Furutani, A. Perali, and LS, Phys. Rev. A **107**, L041302 (2023).

Renormalized superfluid density (II)

The renormalized superfluid density $n_s(\tau = +\infty)$ is obtained by solving the generalized Kosterlitz-Thouless RG equations[8]

$$\begin{aligned} \partial_\tau K^{-1}(\tau) &= 4\pi^3 \Theta(\omega_{\rm R}) y^2(\tau) \\ \partial_\tau y(\tau) &= \left[2 - \pi \Theta(\omega_{\rm R}) K(\tau)\right] y(\tau) \end{aligned} \tag{20}$$

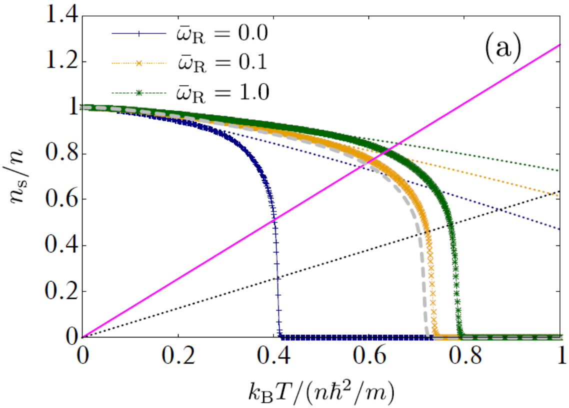
where $K(\tau) = n_{\rm s}(\tau)/T$, with $n_{\rm s}(\tau)$ the superfluid density at the adimensional fictitious time τ, and $y(\tau) = \exp\left[-\mu_{\rm c}(\tau)/T\right]$ is the fugacity, where $\mu_{\rm c}(\tau)$ is the vortex chemical potential at fictitious time τ. Here

$$\Theta(\omega_{\rm R}) = \begin{cases} 1/2 & \text{for} \quad \omega_{\rm R} = 0 \\ 1 & \text{for} \quad \omega_{\rm R} \neq 0 \end{cases} \tag{21}$$

takes into account the quite peculiar formation of "half-vortices" for $\omega_{\rm R} = 0$: one vortex in the hyperfine state a and no vortex in the hyperfine state $3 - a$. For $\omega_{\rm R} \neq 0$ these "half-vortices" are unstable.[9]

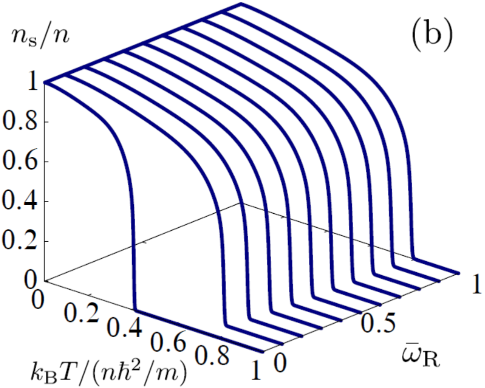
[8] K. Furutani, A. Perali, and LS, Phys. Rev. A **107**, L041302 (2023).
[9] M. Kobayashi, M. Eto, and M. Nitta, Phys. Rev. Lett. **123**, 075303 (2019).

Renormalized superfluid density (III)



Renormalized superfluid fraction n_s/n as a function of the temperature T for $\tilde{g} = mg/\hbar^2 = 0.1$ and $\eta = g_{12}/g = 0$. Here $\bar{\omega}_{\rm R} = \hbar\omega_{\rm R}/(n\hbar^2/m)$. Grey dashed curve: single-component Bose gas. Thin dotted curves: bare superfluid fraction. The thin solid line and thin dotted line stand for $k_{\rm B}T = \pi\hbar^2 n_{\rm s}(T)/(4m)$ and $k_{\rm B}T = \pi\hbar^2 n_{\rm s}(T)/(2m)$ respectively. [**K. Furutani, A. Perali, and LS, Phys. Rev. A 107, L041302 (2023)**].

Renormalized superfluid density (IV)



3D plot of the renormalized superfluid fraction n_s/n as a function of temperature T and adimensional Rabi coupling $\hbar\omega_R/(n\hbar^2/m)$. Also here $\tilde{g} = mg/\hbar^2 = 0.1$ and $\eta = g_{12}/g = 0$. [**K. Furutani, A. Perali, and LS, Phys. Rev. A 107, L041302 (2023)**].

BKT critical temperature (I)

Our RG equations give a modified Nelson-Kosterlitz criterion

$$k_{\rm B} T_{\rm c} = \frac{\pi \hbar^2}{2m} \Theta(\omega_{\rm R}) n_{\rm s}(T_{\rm c}), \tag{22}$$

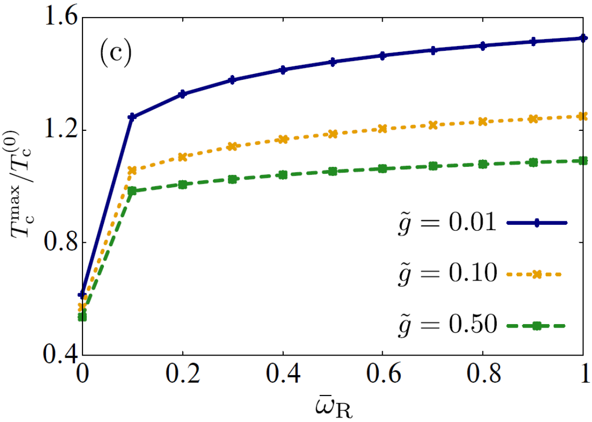
at the BKT critical temperature $T_{\rm c}$, where

$$\Theta(\omega_{\rm R}) = \begin{cases} 1/2 & \text{for} \quad \omega_{\rm R} = 0 \\ 1 & \text{for} \quad \omega_{\rm R} \neq 0 \end{cases} \tag{23}$$

This criterion (22) is consistent with a recent Monte Carlo analysis.[10]

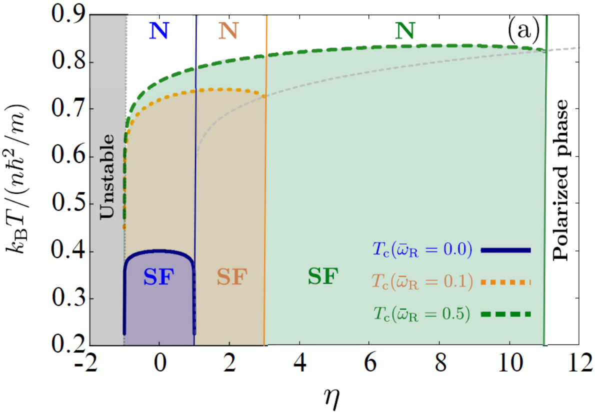
[10] M. Kobayashi, M. Eto, and M. Nitta, Phys. Rev. Lett. **123**, 075303 (2019).

BKT critical temperature (II)



Maximal BKT critical temperature T_c^{max} (for all the values of g_{12}) as a function of the adimensional Rabi coupling $\bar{\omega}_R = \hbar\omega_R/(n\hbar^2/m)$ for three values of $\tilde{g} = mg/\hbar^2 = 0.01, 0.1, 0.5$. Here $T_c^{(0)}$ is the BKT temperature of the single-component case. [**K. Furutani, A. Perali, and LS, Phys. Rev. A 107, L041302 (2023)**].

BKT critical temperature (III)



Phase diagram of the binary Bose mixture and the BKT transition temperature to inter-component coupling $\eta = g_{12}/g$ and Rabi coupling $\bar{\omega}_R = \hbar\omega_R/(n\hbar^2/m)$. SF means "superfluid" and N means "normal". Above the solid vertical lines the system enters in the polarized phase. Gray dotted curve: T_c vs η with ω_R such that $\eta = 1 + 2\hbar\omega_R/(gn)$. [**K. Furutani, A. Perali, and LS, Phys. Rev. A 107, L041302 (2023)**].

First and second sound (I)

We have seen that, according to Landau's two fluid theory[11] of **superfluids**, the total number density n of a system in the superfluid phase can be written as

$$n = n_{\mathrm{s}} + n_{\mathrm{n}} \ , \tag{24}$$

where n_{s} is the superfluid density and n_{n} is the normal density. At the critical temperature T_c one has $n_n = n$ and, correspondingly, $n_{\mathrm{s}} = 0$.

Following Landau, in a **superfluid** a local perturbation excites two wave-like modes - first and second sound - which propagate with velocities c_1 and c_2. These velocities are determined by the positive solutions of the algebraic biquadratic equation

$$c^4 + (v_A^2 + v_L^2)c^2 + v_T^2 v_L^2 = 0 \ . \tag{25}$$

The first sound c_1 is the largest of the two positive roots of Eq. (25) while the second sound c_2 is the smallest positive one.

[11] L.D. Landau, J. Phys. (USSR) **5**, 71 (1941).

First and second sound (II)

In the biquadratic equation there is the adiabatic sound velocity

$$v_A = \sqrt{\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_{\bar{S},V}} \tag{26}$$

with $\bar{S} = S/N$ the entropy per particle, the entropic sound (or Landau) velocity,

$$v_L = \sqrt{\frac{1}{m}\frac{\bar{S}^2}{\left(\frac{\partial \bar{S}}{\partial T}\right)_{N,V}}\frac{n_s}{n_n}} \tag{27}$$

with n_s/n_n the ratio between superfluid and normal density, and the isothermal sound velocity

$$v_T = \sqrt{\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_{T,V}} \,. \tag{28}$$

First and second sound (III)

All the needed thermodynamical quantities can be derived from the Helmholtz free energy[12]

$$F = \frac{1+\eta}{4}\frac{gN^2}{L^2} - \hbar\omega_{\mathrm{R}} N + L^2 k_{\mathrm{B}} T \int \frac{d^2\mathbf{k}}{(2\pi)^2} \Big[\ln\left(1 - e^{-E_k^{(-)}/(k_{\mathrm{B}} T)}\right) + \ln\left(1 - e^{-E_k^{(+)}/(k_{\mathrm{B}} T)}\right) \Big] . \quad (29)$$

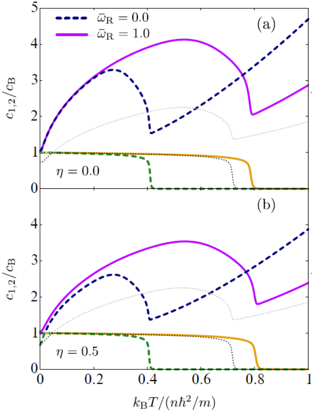
In particular, we have

$$P = -\left(\frac{\partial F}{\partial L^2}\right)_{N,T} , \quad (30)$$

$$\bar{S} = \frac{1}{mN}\left(\frac{\partial F}{\partial T}\right)_{N,L^2} . \quad (31)$$

[12] K. Furutani, A. Tononi, and LS, New J. Phys. **23** 043043 (2021); K. Furutani, A. Perali, and LS, Phys. Rev. A **107**, L041302 (2023).

First and second sound (IV)



First sound and second sound velocities $c_{1,2}$ scaled by the Bogoliubov velocity $c_{\rm B} = \sqrt{gn(1+\eta)/(2m)}$ as a function of the temperature T with $\tilde{g} = mg/\hbar^2 = 0.1$. The thin dotted curves represent $c_{1,2}$ in a single-component Bose gas. Here $\eta = g_{12}/g$ and $\bar{\omega}_{\rm R} = \hbar\omega_{\rm R}/(n\hbar^2/m)$. [**K. Furutani, A. Perali, and LS, Phys. Rev. A 107, L041302 (2023)**].

Conclusions

- We have investigated BKT transition in a Rabi-coupled binary Bose mixture under balanced densities.
- We have found that the renormalized superfluid fraction and the BKT critical temperature are strongly dependent on Rabi coupling and interaction strengths.
- We have also studied the first sound and second sound velocity in this binary Bose mixture by adopting and extending the Landau's two-fluid model.

Acknowledgements

Thank you for your attention!

Main sponsors: INFN Iniziativa Specifica "Quantum" and UNIPD BIRD Project "Ultracold atoms in curved geometries".