Dissipation and Fluctuations in Elongated Josephson Junctions

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Summary

- Modeling an elongated Josephson junction
- Density-phase representation
- Quasi-particle description
- Damped dynamics of the Josephson mode
- The effect of the quantum-thermal noise
- Fluctuations of the Josephson mode
- Conclusions

Modeling an elongated Josephson junction (I)

We start from the following Lagrangian density which consists of two weakly-interacting Bose-Einstein quasi-condensates ($j = 1, 2$) made of atoms with mass m in one dimension

$$
\begin{aligned}
\mathscr{L} &= \sum_{j=1}^{2} \left[i\hbar \psi_j^*(x,t) \partial_t \psi_j(x,t) - \frac{\hbar^2}{2m} |\partial_x \psi_j(x,t)|^2 - \frac{g}{2} |\psi_j(x,t)|^4 \right] \\
&+ \frac{J(x)}{2} \left[\psi_1^*(x,t)\psi_2(x,t) + \psi_2^*(x,t)\psi_1(x,t) \right] \ , \qquad (1)
\end{aligned}
$$

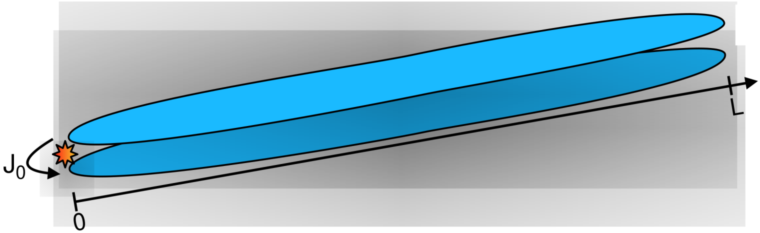
where $\hbar$ is the Planck constant. Here g is the interaction strength for atoms of the same species, $\psi_j(x,t)$ is the complex field of the j-th quasi-condensate, and $J(x)$ is the space dependent tunneling (hopping) coupling.

Modeling an elongated Josephson junction (II)

In the rest of the paper, we assume that

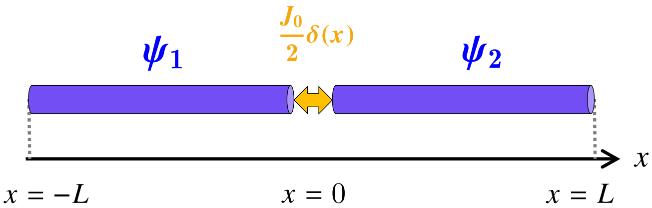
$$\boxed{J(x) = J_0\,\delta(x)}\,. \tag{2}$$

with J_0 a constant tunneling coupling, $x \in [0, L]$ and L the length of the two elongated Bose-Einstein quasi-condensates.



Modeling an elongated Josephson junction (III)

This configuration is however equivalent to a system where one quasi-condensate is confined in the region $[-L, 0]$ while the other quasi-condensate is confined in the region $[0, L]$, and the tunneling barrier is located at $x = 0$.



Density-phase representation (I)

The complex field $\psi_j(x,t)$ of the j-th quasi-condensate can be rewritten by means of the Madelung representation

$$\psi_j(x,t) = \sqrt{\rho_j(x,t)}\, e^{i\phi_j(x,t)}, \tag{3}$$

where $\rho_j(x,t) = |\psi_j(x,t)|^2$ is its local number density and $\phi_j(x,t)$ is the local phase of the j-th quasi-condensate. Substituting the expression of Eq. (3) into the Lagrangian in Eq. (1), we obtain

$$\begin{aligned} \mathscr{L} &= \sum_{j=1}^{2} \left[\frac{i\hbar}{2}\partial_t \rho_j - \hbar \rho_j \partial_t \phi_j - \frac{\hbar^2}{2m}\left[\frac{1}{4\rho_j}(\partial_x \rho_j)^2 + \rho_j (\partial_x \phi_j)^2 \right] - \frac{g}{2}\rho_j^2 \right] \\ &+ J_0 \delta(x) \sqrt{\rho_1 \rho_2} \cos(\phi_1 - \phi_2)\,. \end{aligned} \tag{4}$$

Density-phase representation (II)

A compact description of the system is reached when we introduce the relative phase and the population imbalance

$$\begin{aligned} \phi(x,t) &= \phi_1(x,t) - \phi_2(x,t) \ , & (5) \\ \zeta(x,t) &= \frac{\rho_1(x,t) - \rho_2(x,t)}{2\bar{\rho}}, & (6) \end{aligned}$$

with $\bar{\rho} = N/L$ the average atomic density with N the number of bosons. In this way we obtain a new Lagrangian density[1]

$$\begin{aligned} \mathscr{L} &= -\hbar\bar{\rho}\zeta\dot{\phi} - \frac{\hbar^2\bar{\rho}}{4m}(\partial_x\phi)^2 - g(\bar{\rho}^2 + \bar{\rho}^2\zeta^2) \\ &+ J(x)\bar{\rho}\sqrt{1-\zeta^2}\cos(\phi) & (7) \end{aligned}$$

in which we neglected space derivatives of the population imbalance.

[1] A. Tononi, F. Toigo, S. Wimberger, A. Cappellaro, and LS, New J. Phys. **22**, 073020 (2020).

Density-phase representation (III)

We work with small values of the population imbalance, such that $|\zeta(x,t)| \ll 1$, and in the Josephson regime $2g\bar{\rho} \gg J_0\delta(x)$. Under these conditions the Euler-Lagrange equation of motion of $\zeta(x,t)$ is quite simple

$$\zeta(x,t) = -\frac{\hbar\dot{\phi}(x,t)}{2g\bar{\rho}} \ . \tag{8}$$

Inserting it into the previous Lagrangian density we obtain a Lagrangian density[2] for the relative phase field $\phi(x,t)$:

$$\boxed{\mathscr{L} = \frac{\hbar}{4g}\dot{\phi}^2 - \frac{\hbar^2\bar{\rho}}{4m}(\partial_x\phi)^2 + J_0\delta(x)\,\bar{\rho}\,\cos(\phi)} \ . \tag{9}$$

This is the Lagrangian density of the so-called boundary sine-Gordon model.[3]

[2] A. Tononi, F. Toigo, S. Wimberger, A. Cappellaro, and LS, New J. Phys. **22**, 073020 (2020).

[3] P. Fendley, F. Lesage, and H. Saleur, J. Stat. Phys. **85**, 211 (1996).

Quasi-particle description (I)

We now introduce a quasi-particle description for the phase field $\phi(x,t)$, based on the following mode expansion

$$\phi(x,t) = \frac{1}{\sqrt{L}} \sum_{n=0}^{+\infty} q_n(t)\Phi_n(x), \qquad (10)$$

where $q_n(t)$ are coordinates, and $\Phi_n(x)$ are real eigenfunctions satisfying $-\hbar^2/(2m)\,\partial_x^2\Phi_n(x) = \epsilon_n\Phi_n(x)$ where $\epsilon_n = \hbar^2 k_n^2/(2m)$ and $k_n = \pi n/L$, and constituting an orthonormal basis $\int_0^L \Phi_n(x)\Phi_m(x)\,dx = \delta_{n,m}$. Inserting Eq. (10) into Eq. (9) gives

$$\mathcal{L} = \int_0^L \mathscr{L}\,dx = \frac{M}{2}\sum_{n=0}^{+\infty} \dot{q}_n^2 - \frac{M}{2}\sum_{n=0}^{+\infty} \omega_n^2 q_n^2 + J_0\bar{\rho}\cos\left(\frac{1}{L}\sum_{n=0}^{+\infty} q_n\right) . \qquad (11)$$

Here we have defined

$$M = \frac{\hbar^2}{2gL} \qquad \textit{and} \qquad \omega_n = c_s\,k_n\ , \qquad (12)$$

where $c_s = \sqrt{\bar{\rho}g/m}$ is the speed of sound.

Quasi-particle description (II)

We identify the Josephson mode as the field at $x = 0$, namely

$$\phi(x=0,t) = \frac{1}{L}\sum_{n=0}^{+\infty} q_n(t) = \frac{Q_0(t)}{L} = \phi_0(t) \;, \tag{13}$$

introducing a new "collective" Josephson variable $Q_0(t)$. After a Legendre transformation from Eq. (11) we obtain the Hamiltonian

$$\boxed{\mathcal{H} = \frac{P_0^2}{2M} + \sum_{n=1}^{+\infty}\left[\frac{(P_0+p_n)^2}{2M} + \frac{1}{2}M\omega_n^2 q_n^2\right] - J_0\bar{\rho}\cos(\frac{Q_0}{L})} \;, \tag{14}$$

which shows a coupling between the linear momentum $P_0(t)$ of the Josephson mode with generalized coordinate $Q_0(t)$ and the linear momenta $p_n(t)$ of the sound modes with generalized coordinates $q_n(t)$ $(n \neq 0)$. This Hamiltonian[4] is similar (but ont equal) to the Caldeira-Leggett one[5].

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[4] J. Polo *et al.*, Phys. Rev. Lett. **121**, 090404 (2018).
[5] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981).

Damped dynamics of the Josephson mode (I)

From the Hamilton equations of (14) we obtain

$$\boxed{\ddot{\phi}_0(t) + \gamma_0\,\dot{\phi}_0(t) + \Omega_0^2 \sin(\phi_0(t)) = \xi_\phi(t)}\,, \tag{15}$$

where

$$\Omega_0 = \sqrt{\frac{\bar{\rho} J_0}{M L^2}} \tag{16}$$

is the Josephson frequency,

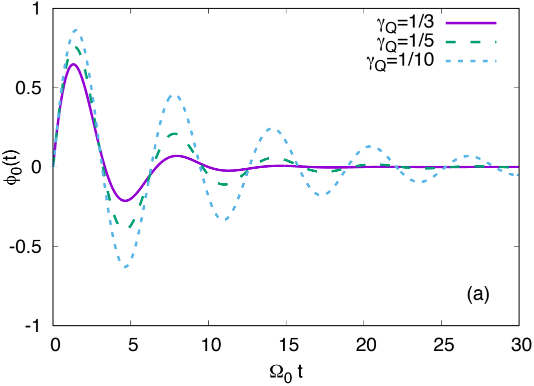
$$\gamma_0 = \frac{\bar{\rho} J_0}{M L c_s} \tag{17}$$

is the damping coefficient due to the phonon bath and

$$\xi_\phi(t) = -\sum_{n=1}^{+\infty} \left[\omega_n^2 \cos(\omega_n t) \frac{q_n(0)}{L} + \omega_n \sin(\omega_n t) \frac{\dot{q}_n(0)}{L} \right] \tag{18}$$

is a noise due to the initial conditions of the phonon bath.

Damped dynamics of the Josephson mode (II)



Relative phase dynamics of the Josephson mode without the effect of the noise ($\xi_\phi(t) = 0$). Time evolution of the Josephson relative phase $\phi_0(t)$. [6] Here we consider the underdamped case with $\gamma_Q = \gamma_0/(2\Omega_0) < 1$. Initial conditions: $\phi_0(0) = 0$ and $\dot{\phi}_0(0)/\Omega_0 = 1$.

[6] F. Binanti, K. Furutani, and LS, Phys. Rev. A **103**, 063309 (2021).

The effect of the quantum-thermal noise (I)

We have seen that the noise $\xi_\phi(t)$ crucially depends on the initial conditions $q_n(0)$ and $\dot{q}_n(0) = p_n(0)/M$ of the phonon bath, Eq. (18).
The phonon bath is characterized by infinite harmonic oscillators with Hamiltonian

$$\mathcal{H}_{\rm B} = \sum_{n=1}^{+\infty} \left[\frac{p_n^2}{2M} + \frac{M\omega_n^2}{2} q_n^2 \right] \tag{19}$$

which we use to evaluate the ensemble average at temperature T where $q_n(t)$ and $p_n(t)$ are quantum operators.
Thus, for a generic observabile $A(t)$ its quantum-thermal average reads

$$\langle A(t) \rangle = \frac{\text{Tr}\left[A(t)\, e^{-\beta \mathcal{H}_{\rm B}} \right]}{\text{Tr}\left[e^{-\beta \mathcal{H}_{\rm B}} \right]} \,, \tag{20}$$

where $\beta = 1/(k_{\rm B} T)$ with $k_{\rm B}$ the Boltzmann constant and T the absolute temperature of the bath of oscillators.

The effect of the quantum-thermal noise (II)

Clearly, one finds that

$$\langle q_n(t)^2\rangle = \frac{\hbar}{M\omega_n}\left(\frac{1}{e^{\beta\hbar\omega_n}-1}+\frac{1}{2}\right) \tag{21}$$

$$\langle p_n(t)^2\rangle = M\hbar\omega_n\left(\frac{1}{e^{\beta\hbar\omega_n}-1}+\frac{1}{2}\right) \tag{22}$$

It is then possible to prove[7] that the noise $\xi_\phi(t)$ is such that

$$\langle \xi_\phi(t)\rangle = 0 \, , \tag{23}$$

$$\langle \xi_\phi(t)\xi_\phi(t')\rangle = \sum_{n=1}^{+\infty}\frac{\hbar\omega_n^3}{2ML^2}\left[\coth(\frac{\beta\hbar\omega_n}{2})\cos\left[\omega_n(t-t')\right] - i\sin\left[\omega_n(t-t')\right]\right] . \tag{24}$$

[7] Note that

$$\frac{1}{e^x-1}+\frac{1}{2}=\frac{1}{2}\coth(\frac{x}{2}).$$

The effect of the quantum-thermal noise (III)

Considering the linearized damped equation with noise

$$\ddot{\phi}_0(t) + \gamma_0 \, \dot{\phi}_0(t) + \Omega_0^2 \, \phi_0(t) = \xi_\phi(t) \ , \tag{25}$$

because $\langle \xi_\phi(t) \rangle = 0$ one finds immediately

$$\frac{d^2}{dt^2} \langle \phi_0(t) \rangle + \gamma_0 \, \frac{d}{dt} \langle \phi_0(t) \rangle + \Omega_0^2 \, \langle \phi_0(t) \rangle = 0 \ . \tag{26}$$

Thus, the time evolution of $\langle \phi_0(t) \rangle$ is independent on the noise $\xi_\phi(t)$.

The dynamics of $\langle \phi_0(t) \rangle$ is the so-called mean-field solution, that we have previously analyzed considering the case of damping in the absence of noise.

The effect of the quantum-thermal noise (IV)

Morover, one finds that Eq. (25) admits a formal solution[8]

$$\phi_0(t) = e^{-\gamma_Q \Omega_0 t} \frac{\sin(\gamma_J \Omega_0 t)}{\gamma_J} + \int_0^t dt' \chi(t - t') \xi_\phi(t'), \tag{27}$$

where

$$\chi(t - t') = \frac{2}{\omega_D} e^{-\gamma_Q \Omega_0 (t - t')} \sin\left[\frac{\omega_D}{2}(t - t')\right] \theta(t - t'), \tag{28}$$

with $\theta(t)$ the Heaviside step function and

$$\omega_D = \sqrt{4\Omega_0^2 - \gamma_0^2}\,. \tag{29}$$

[8] F. Binanti, K. Furutani, and LS, Phys. Rev. A **103**, 063309 (2021).

Fluctuations of the Josephson mode (I)

We can then calculate[9] the variance $\Delta\phi_0(t)$, i.e. the quadratic fluctuation, of the Josephson relative phase $\phi_0(t) = \phi(x = 0, t)$, defined as

$$\Delta\phi_0(t)^2 = \langle\phi_0(t)^2\rangle - \langle\phi_0(t)\rangle^2 \; . \tag{30}$$

For the sake of completeness we report also the variance $\Delta\zeta_0(t)$ of the Josephson population imbalance $\zeta_0(t) = \zeta(x = 0, t)$, defined as

$$\Delta\zeta_0(t)^2 = \langle\zeta_0(t)^2\rangle - \langle\zeta_0(t)\rangle^2 \; . \tag{31}$$

In the high-temperature regime $k_B T \gg \hbar\Omega_0$ we find analytical expressions for the asymptotic values of the variations:

$$\Delta\phi_0(\infty) = \frac{1}{\Omega_0}\sqrt{\frac{k_{\rm B} T}{2ML^2}} \; , \tag{32}$$

$$\Delta\zeta_0(\infty) = \sqrt{\frac{M k_{\rm B} T}{2\hbar^2 \bar{\rho}^2}} \; . \tag{33}$$

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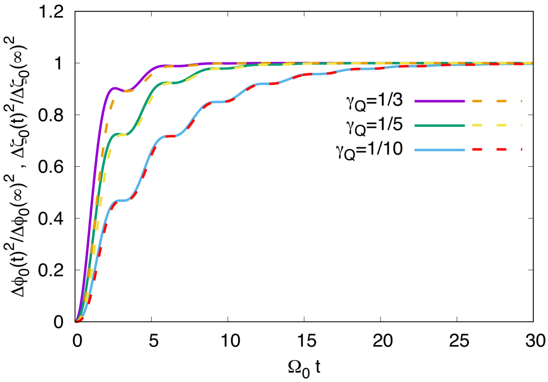
[9] In general, the numerical results depend on a ultraviolet cutoff $k_{max} = \pi\bar{\rho}$, where $\bar{\rho} = N/L$ is the average number density. We use $\bar{\rho} = 10^3\ \mu\text{m}^{-1}$.

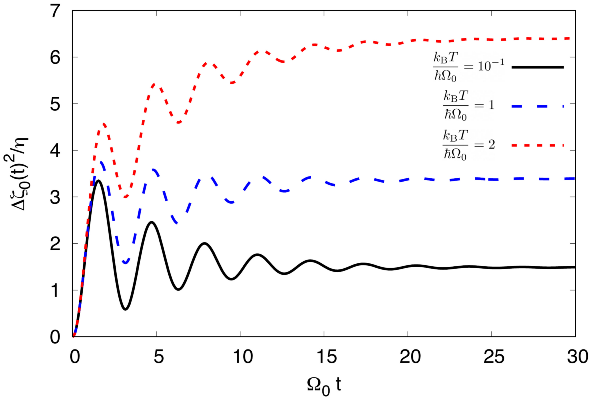
Fluctuations of the Josephson mode (II)



Relative phase variance $\Delta\phi_0(t)^2$ (solid curves) and population imbalance variance $\Delta\zeta_0(t)^2$ (dashed curves) of the Josephson mode as a function of time t.[10] The curves correspond to three different underdamped regimes with a high temperature $k_{\rm B}T/(\hbar\Omega_0) = 10$. The variances are nomalized by their asymptotic values. Here $\gamma_Q = \gamma_0/(2\Omega_0) < 1$.

[10] F. Binanti, K. Furutani, and LS, Phys. Rev. A **103**, 063309 (2021).

Fluctuations of the Josephson mode (III)



Variance $\Delta\zeta_0(t)^2$ of the Josephson population imbalance as a function of time t for three values of the temperature T of the bath of phonons.[11] The damping coefficient γ_Q is set to be $\gamma_Q = 1/10$. Ω_0 is the Josephson frequency while $\eta = M\Omega_0/(\hbar\bar{\rho}^2)$.

[11] F. Binanti, K. Furutani, and LS, Phys. Rev. A **103**, 063309 (2021).

Conclusions

- We have investigated the dynamics of bosonic atoms in elongated Josephson junctions.[12]
- We have found that these systems are characterized by an intrinsic coupling between the Josephson mode of macroscopic quantum tunneling and the sound modes.
- This coupling of Josephson and sound modes gives rise to a damped and stochastic Langevin dynamics for the Josephson degree of freedom.
- The time evolution of the Josephson fluctuations exhibits a thermalization to constant values after a transient characterized by an oscillating dynamics.
- We have recently adopted similar techniques to study the effect of quantum and thermal fluctuations in superconducting Josephson junctions.[13]

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[12] F. Binanti, K. Furutani, and LS, Phys. Rev. A **103**, 063309 (2021).
[13] K. Furutani and LS, arXiv:2104.14211, Phys. Rev. B, in press.

Acknowledgements

Thank you for your attention!

Slides online: http://materia.dfa.unipd.it/salasnich/talk-lphys21.pdf