

# Bose-Einstein Condensation on the Surface of a Sphere

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Online Workshop on Quantum Bubble Physics,  
Leibniz University Hannover, April 7, 2022

Research done in collaboration with A. Tononi, F. Cinti, and A. Pelster.

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# Introduction

**Bose-Einstein condensates** (BECs) made of ultracold alkali-metal atoms under **microgravity** were achieved dropping the BEC down a 146-meter-long drop chamber<sup>1</sup>, but also rocketing the BEC and conducting experiments during in-space flight.<sup>2</sup>



In 2020 a BEC in harmonic trap<sup>3</sup> has been observed with the **NASA's Cold Atom Laboratory** onboard of the **International Space Station**. Moreover, in 2021 the same team has reported the observation of ultracold atomic bubbles.<sup>4</sup>

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<sup>1</sup>T. van Zoest, et al., Science **328**, 1540 (2010)

<sup>2</sup>D. Becker et al., Nature **562**, 391 (2018).

<sup>3</sup>D.C. Aveline et al., Nature **582**, 193 (2020).

<sup>4</sup>R.A. Carollo et al., e-preprint arXiv:2108.05880.

# Bose gas on the surface of a sphere

Our theoretical study of a Bose gas on the surface of a sphere is triggered by the experimental confinement the atoms on a bubble trap,<sup>5</sup> which needs microgravity conditions.<sup>6</sup>

The energy of a particle of mass  $m$  moving on the surface of a sphere of radius  $R$  is quantized according to the formula

$$\epsilon_l = \frac{\hbar^2}{2mR^2} l(l+1), \quad (1)$$

where  $\hbar$  is the reduced Planck constant and  $l = 0, 1, 2, \dots$  is the **integer quantum number** of the angular momentum. This energy level has the degeneracy  $2l + 1$  due to the magnetic quantum number  $m_l = -l, -l + 1, \dots, l - 1, l$  of the third component of the angular momentum.

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<sup>5</sup>B. M. Garraway and H. Perrin, J. Phys. B **49**, 172001 (2016).

<sup>6</sup>E.R. Elliott et al., npj Microgravity **4**, 16 (2018); R.A. Carollo et al., e-preprint arXiv:2108.05880.

# Non-interacting bosons: critical temperature (I)

In quantum statistical mechanics the total number  $N$  of **non-interacting bosons** moving on the surface of a sphere and at equilibrium with a thermal bath of absolute temperature  $T$  is given by

$$N = \sum_{l=0}^{+\infty} \frac{2l+1}{e^{(\epsilon_l - \mu)/(k_B T)} - 1}, \quad (2)$$

where  $k_B$  is the Boltzmann constant and  $\mu$  is the chemical potential. In the Bose-condensed phase, we can set<sup>7</sup>  $\mu = 0$  and

$$N = N_0 + \sum_{l=1}^{+\infty} \frac{2l+1}{e^{\epsilon_l/(k_B T)} - 1}, \quad (3)$$

where  $N_0$  is the number of bosons in the lowest single-particle energy state, i.e. the **number of bosons in the Bose-Einstein condensate (BEC)**.

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<sup>7</sup>For details, see Martina Russo, BSc thesis, Supervisor: LS, Univ. of Padova (2019).

# Non-interacting bosons: critical temperature (II)

Within the semiclassical approximation, where  $\sum_{l=1}^{+\infty} \rightarrow \int_1^{+\infty} dl$ , the previous equation becomes

$$n = n_0 + \frac{mk_B T}{2\pi\hbar^2} \left( \frac{\hbar^2}{mR^2 k_B T} - \ln \left( e^{\hbar^2/(mR^2 k_B T)} - 1 \right) \right), \quad (4)$$

where  $n = N/(4\pi R^2)$  is the 2D number density and  $n_0 = N_0/(4\pi R^2)$  is the 2D condensate density.

At the critical temperature  $T_{BEC}$ , where  $n_0 = 0$ , one then finds<sup>8</sup>

$$k_B T_{BEC} = \frac{\frac{2\pi\hbar^2}{m} n}{\frac{\hbar^2}{mR^2 k_B T_{BEC}} - \ln \left( e^{\hbar^2/(mR^2 k_B T_{BEC})} - 1 \right)}. \quad (5)$$

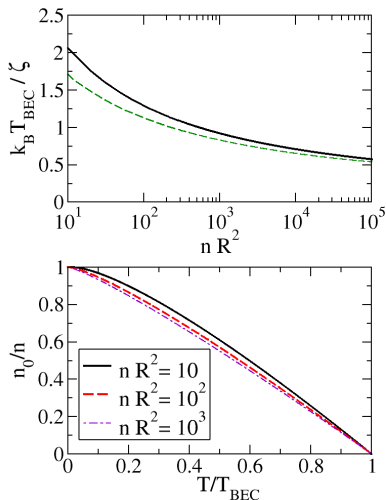
As expected, in the limit  $R \rightarrow +\infty$  one gets  $T_{BEC} \rightarrow 0$ , in agreement with the Mermin-Wagner theorem.<sup>9</sup> However, for any finite value of  $R$  the critical temperature  $T_{BEC}$  is larger than zero.

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<sup>8</sup>A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).

<sup>9</sup>N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

# Non-interacting bosons: critical temperature (III)



**Top panel:**  $T_{BEC}$  vs  $nR^2$ , with  $\zeta = \hbar^2 n/m$ . Solid line: semiclassical approximation (solid line); dashed line: numerical evaluation of the sum.

**Bottom panel:** condensate fraction  $n_0/n$  vs temperature  $T/T_{BEC}$ .

# Interacting bosons: thermodynamics (I)

We now consider a system of **interacting bosons** on the surface of a sphere of radius  $R$  and **contact interaction of strength  $g$** .<sup>10</sup>

Adopting functional integration the partition function  $\mathcal{Z}$  reads

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}, \psi] e^{-\frac{S[\bar{\psi}, \psi]}{\hbar}}, \quad (6)$$

where, by using  $\beta = 1/(k_B T)$  with  $T$  the absolute temperature,

$$S[\bar{\psi}, \psi] = \int_0^{\beta\hbar} d\tau \int_0^{2\pi} d\varphi \int_0^\pi \sin(\theta) d\theta R^2 \mathcal{L}(\bar{\psi}, \psi) \quad (7)$$

is the Euclidean action and, with  $\hat{L}$  is the angular momentum operator,

$$\mathcal{L} = \bar{\psi}(\theta, \varphi, \tau) \left( \hbar \partial_\tau + \frac{\hat{L}^2}{2mR^2} - \mu \right) \psi(\theta, \varphi, \tau) + \frac{g}{2} |\psi(\theta, \varphi, \tau)|^4 \quad (8)$$

is the Euclidean Lagrangian of the bosonic field  $\psi(\theta, \phi, \tau)$ , which depends on the spherical angles  $\theta$  and  $\phi$  and on the imaginary time  $\tau$ .

<sup>10</sup>A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).



# Interacting bosons: thermodynamics (II)

The condensate phase is introduced with the Bogoliubov shift

$$\psi(\theta, \varphi, \tau) = \psi_0 + \eta(\theta, \varphi, \tau), \quad (9)$$

where the real field configuration  $\psi_0$  describes the condensate component. By substituting this field parametrization and keeping only second order terms in the field  $\eta$  we rewrite the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g \quad (10)$$

with  $\mathcal{L}_0 = -\mu\psi_0^2 + g\psi_0^4/2$ .

We use the following decomposition of the complex fluctuation field  $\eta(\theta, \varphi, \tau)$

$$\eta(\theta, \varphi, \tau) = \sum_{\omega_n} \sum_{l=1}^{\infty} \sum_{m_l=-l}^l \frac{e^{-i\omega_n\tau}}{R} \mathcal{Y}_{m_l}^l(\theta, \varphi) \eta(l, m_l, \omega_n), \quad (11)$$

where  $\omega_n = 2\pi n/(\hbar\beta)$  are the Matsubara frequencies, and we introduce the orthonormal basis of the spherical harmonics  $\mathcal{Y}_{m_l}^l(\theta, \phi)$ .

# Interacting bosons: thermodynamics (III)

After some analytical calculations, at the Gaussian level the grand potential

$$\Omega = -\frac{1}{\beta} \ln(\mathcal{Z}) \simeq -\frac{1}{\beta} (\ln(\mathcal{Z}_0) + \ln(\mathcal{Z}_g)) \quad (12)$$

is given by

$$\begin{aligned} \Omega(\mu, \psi_0^2) &= 4\pi R^2 \left( -\mu \psi_0^2 + g \psi_0^4 / 2 \right) + \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m_l=-l}^l E_l(\mu, \psi_0^2) \\ &+ \frac{1}{\beta} \sum_{l=1}^{\infty} \sum_{m_l=-l}^l \ln(1 - e^{-\beta E_l(\mu, \psi_0^2)}) \end{aligned} \quad (13)$$

where

$$E_l(\mu, \psi_0^2) = \sqrt{(\epsilon_l - \mu + 2g\psi_0^2)^2 - g^2\psi_0^4} \quad (14)$$

is the excitation spectrum of the interacting system, with  $\epsilon_l = \hbar^2 l(l+1)/(2mR^2)$  the single-particle energy.

# Interacting bosons: thermodynamics (IV)

The condensate number density  $n_0$  of the system is given by

$$n_0 = \psi_0^2, \quad (15)$$

where we fix the value of the order parameter  $\psi_0$  with the condition

$$\frac{\partial \Omega(\mu, \psi_0^2)}{\partial \psi_0} = 0. \quad (16)$$

Notice that from this formula we get  $n_0$  as a function of  $\mu$ .  
The total number density of the system is instead given by

$$n = -\frac{1}{4\pi R^2} \frac{\partial \Omega(\mu, n_0(\mu))}{\partial \mu}. \quad (17)$$

At the lowest order of a perturbative scheme,<sup>11</sup> where  $\psi_0$  is obtained from the mean-field equation  $\frac{\partial \Omega_0(\mu, \psi_0^2)}{\partial \psi_0} = 0$ , we get  $\psi_0 \simeq \sqrt{\mu/g}$  and

$$E_l \simeq E_l^B = \sqrt{\epsilon_l(\epsilon_l + 2\mu)}. \quad (18)$$

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<sup>11</sup>H. Kleinert, S. Schmidt, and A. Pelster, Phys. Rev. Lett. **93**, 160402 (2004).

# Interacting bosons: thermodynamics (V)

Within this perturbative scheme<sup>12</sup> from the previous equations we obtain<sup>13</sup> the **BEC critical temperature**

$$k_B T_{BEC} = \frac{\frac{2\pi\hbar^2 n}{m} - \frac{gn}{2}}{\frac{\hbar^2}{2mR^2 k_B T_{BEC}} \left( 1 + \sqrt{1 + \frac{2gmnR^2}{\hbar^2}} \right) - \ln \left( e^{\frac{\hbar^2}{mR^2 k_B T_{BEC}}} \sqrt{1 + \frac{2gmnR^2}{\hbar^2}} - 1 \right)}, \quad (19)$$

where the condensate density  $n_0$  is zero.

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<sup>12</sup>H. Kleinert, S. Schmidt, and A. Pelster, Phys. Rev. Lett. **93**, 160402 (2004).

<sup>13</sup>A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).

# Superfluid density (I)

Adopting the Landau formula for the normal density in a superfluid,<sup>14</sup> we calculate the **superfluid density**  $n_s(T)$  as

$$n_s = n - \frac{1}{k_B T} \int_1^{+\infty} \frac{dl (2l+1)}{4\pi R^2} \frac{\hbar^2(l^2 + l)}{2mR^2} \frac{e^{E_l^B/(k_B T)}}{(e^{E_l^B/(k_B T)} - 1)^2} . \quad (20)$$

Moreover, applying the **Kosterlitz-Nelson criterion**<sup>15</sup> we evaluate numerically the **Berezinskii-Kosterlitz-Thouless critical temperature**  $T_{BKT}$  of the superfluid-normal transition induced by the proliferation of quantized vortices.<sup>16</sup>

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<sup>14</sup>L. Landau, Phys. Rev. **60**, 356 (1941); E.M. Lifshitz and L. P. Pitaevskii, Statistical Physics: Theory of the Condensed State, Course of Theoretical Physics, Vol. 9 (Butterworth-Heinemann, 1980).

<sup>15</sup>D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. **39**, 1201 (1977).

<sup>16</sup>V.L. Berezinskii, Sov. Phys. JETP **34** 610 (1971); J.M. Kosterlitz and D.J. Thouless, Journal of Physics C: Solid State Physics **6** 1181 (1973).

# Superfluid density (II)

In our problem of interacting bosons on the surface of a sphere, we determine the **critical temperature**  $T_{BKT}$  by using the Nelson-Kosterlitz criterion<sup>17</sup>:

$$k_B T_{BKT} = \frac{\pi \hbar^2}{2m} n_s(T_{BKT}^-). \quad (21)$$

However, for the sake of simplicity we use the bare superfluid density  $n_s(T)$  instead of the renormalized one.

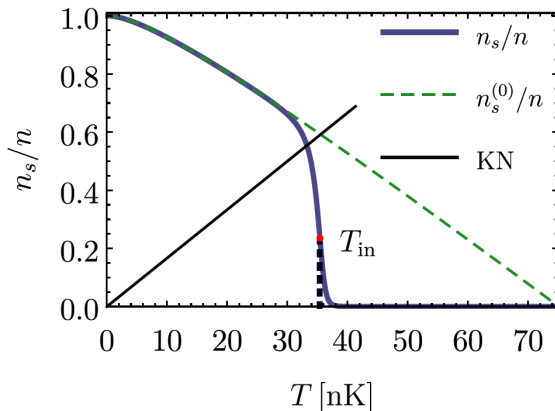
In a very recent paper<sup>18</sup> we have instead used the renormalized superfluid density to determine  $T_{BKT}$  by solving the Kosterlitz-Thouless renormalization group equations.

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<sup>17</sup>D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. **39**, 1201 (1977).

<sup>18</sup>A. Tononi, A. Pelster, and LS, Phys. Rev. Research **4**, 013122 (2022).

# Superfluid density (III)



The bare superfluid density  $n_s^{(0)}$  overestimates the renormalized one  $n_s$ . However, the renormalized superfluid fraction  $n_s/n$  of a shell-shaped superfluid does not display an abrupt jump, but vanishes smoothly around the temperature  $T_{in}$  of the inflection point. Adapted from A. Tononi, A. Pelster, and LS, Phys. Rev. Research **4**, 013122 (2022).

# Phase diagram for bosons on the surface of a sphere (I)

We now analyze the phase diagram of the gas of bosons on the surface of a sphere by using the plane  $(gm/\hbar^2, k_B T/\zeta)$ , where  $gm/\hbar^2$  is the adimensional interaction strength of bosons and  $k_B T/\zeta$  is the adimensional temperature with  $\zeta = \hbar^2 n/m$ .

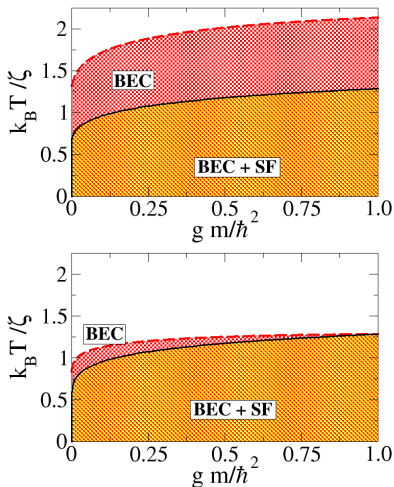
Within the approximations adopted, depending on the values of  $gm/\hbar^2$ ,  $k_B T/\zeta$ , but also  $nR^2$ , the system can show:

- coexistence of condensation and superfluidity (BEC+SF);
- superfluidity in the absence of condensation (SF);
- Bose-Einstein condensation in the absence of superfluidity (BEC).

In the thermodynamic limit, i.e.  $nR^2 \rightarrow +\infty$ , the BEC region shrinks to zero.

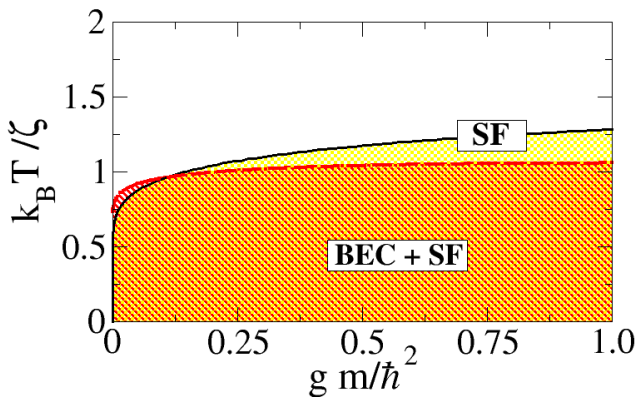


# Phase diagram for bosons on the surface of a sphere (II)



Phase diagram of the bosonic system for  $nR^2 = 10^2$  (**upper panel**) and  $nR^2 = 10^4$  (**lower panel**). Here  $\zeta = \hbar^2 n / m$ . Adapted from A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).

# Phase diagram for bosons on the surface of a sphere (III)



Phase diagram of the bosonic system for  $nR^2 = 10^5$ . Here  $\zeta = \hbar^2 n / m$ .  
Adapted from A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).

# Conclusions (I)

- Triggered by recent achievements of space-based BECs under microgravity and bubble traps, which confine atoms on a thin shell, we have investigated<sup>19</sup> **BEC on the surface of a sphere** finding:
  - BEC critical temperature for non-interacting bosons;
  - BEC thermodynamics, superfluid density, and BEC and BKT critical temperatures for interacting bosons.
- In a recent paper<sup>20</sup>, we have instead analyzed **BEC on the surface of an ellipsoid** for realistic bubble-trap parameters calculating:
  - BEC critical temperature both non-interacting and interacting bosons;
  - the free expansion of the hollow Bose condensate.

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<sup>19</sup>A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).

<sup>20</sup>A. Tononi, F. Cinti, and LS, Phys. Rev. Lett. **125**, 010402 (2020).

# Conclusions (II)

- In a very recent paper<sup>21</sup> we have analyzed in detail the BKT phase transition for a **BEC on the surface of a sphere** calculating the renormalized superfluid density of the system by deriving and solving generalized Kosterlitz-Thouless renormalization group equations.
- See the next talk of **Andrea Tononi** about the 2D equation of state and the relationship between the 2D interaction strength  $g$  and the 2D s-wave scattering length  $a_s$ .<sup>22</sup>

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<sup>21</sup>A. Tononi, A. Pelster, and LS, Phys. Rev. Research **4**, 013122 (2022).

<sup>22</sup>A. Tononi, Phys. Rev. A **105**, 023324 (2022).

# Open problems

- The surface of a sphere has a constant curvature while the surface of an ellipsoid does not have a constant curvature. Does a locally-varying curvature affect the quantum-thermal properties of a Bose gas constrained to move on the surface of an ellipsoid?
- For a particle constrained on a curve it appears a quantum-curvature potential<sup>23</sup>

$$U_{QC}(s) = -\frac{\hbar^2 \kappa(s)^2}{8m} ,$$

where  $\kappa(s)$  is the local geodesic curvature of the curve and  $s$  is the curvilinear abscissa (arclength) along the curve.

- Similarly, also for a particle constrained on a surface it appears a quantum-curvature potential.<sup>24</sup> In the case of the surface of an ellipsoid this quantum-curvature potential could strongly affect the quantum-thermal properties of a Bose gas.

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<sup>23</sup>L. Salasnich, Bose-Einstein condensate in an elliptical waveguide, SciPost Phys. Core **5**, 015 (2022).

<sup>24</sup>N.S. Moller, F.E.A. dos Santos, V.S. Bagnato, and A. Pelster, New. J. Phys. **22**, 063059 (2020).

# Acknowledgements

**Thank you for your attention!**



There is nothing in the intellect that never existed before in the senses  
(John Locke)



... except the intellect itself (Gottfried Wilhelm Leibniz).