Interaction between gravitational waves and trapped Bose-Einstein condensates

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Interdisciplinary Symposium
"Complexity of the Cosmos",
Gran Sasso Science Institute, L'Aquila, October 3, 2024

Research done in collaboration with Alessio Perodi

Summary

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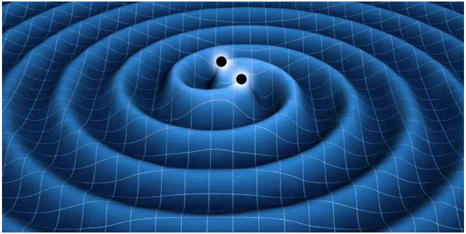
Introduction (I)

Nowadays gravitational waves (GWs) are detected with **very long interferometers**, such as LIGO and Virgo.[1]



In order to develop more compact tools to investigate multimessanger astronomy, it has been suggested[2] that phonons (sound waves) in Bose-Einstein condensates (BECs) can be induced by GWs.

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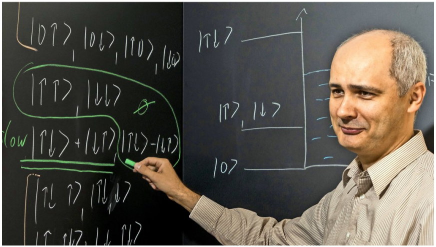
[1] B. P. Abbott et al., Phys. Rev. Lett. **116**, 061102 (2016).

[2] C. Sabín, D.E. Bruschi, M. Ahmadi, and I. Fuentes, New J. Phys. **16**, 085003 (2014); M. P. G. Robbins, N. Afshordi, A.O. Jamison, and R. B. Mann, Class. Quantum Grav. **39**, 175009 (2022).

Introduction (II)

Here we analyze some consequences of a "quantum-information oriented" proposal[3] of Ralf Schützhold (TU Dresden),



who studied the quantum many-body wavefunction of a BEC under the effect of a GW.

[3] R. Schützhold, Phys. Rev. D **98**, 105019 (2018).

Gravitational waves (I)

GWs are perturbation of the metric $g_{\mu\nu}$ of the spacetime and the equations which describe them are obtained starting from the Einstein field equations[4]

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1}$$

We impose the weak field condition

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

is the flat Minkowski metric and $|h_{\mu\nu}| \ll 1$.

[4] R.M. Wald, General Relativity (Chicago Univ. Press, 1984).

Gravitational waves (II)

Proceeding with the calculations one gets the following result

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) h_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu} \tag{4}$$

which are indeed the GW equations, with $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$.
Eq. (5) are not easily solvable analytically, primarily due to the presence of the energy-momentum tensor $T_{\mu\nu}$. Therefore, let us consider the simplest case where $T_{\mu\nu} = 0$, i.e.

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) h_{\mu\nu} = 0 \tag{5}$$

In this case the GW solutions are plane waves.

Gravitational waves (III)

From now on, we will only consider gravitational waves $h_{\mu\nu}$ propagating along the z axis, and such that

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & -h & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (6)$$

which induce the following modification of the spacetime interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + (1+h)dx^2 + (1-h)dy^2 + dz^2 \qquad (7)$$

where h depends only on z and t, i.e. $h = h(z,t)$.

Bose-Einstein condensates (I)

Let us now consider non-relativistic identical particles of mass m in flat spacetime. The quantum-field-theory Hamiltonian is given by[5]

$$\begin{aligned} \hat{H}_{\rm flat} &= \int d^3\mathbf{r}\, \hat{\psi}^+(\mathbf{r},t) \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) \right] \hat{\psi}(\mathbf{r},t) \\ &+ \frac{1}{2}\int d^3\mathbf{r}\, d^3\mathbf{r}'\, \hat{\psi}^+(\mathbf{r},t)\hat{\psi}^+(\mathbf{r}',t) V(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}',t)\hat{\psi}(\mathbf{r},t) \end{aligned} \quad (8)$$

where $\hat{\psi}(\mathbf{r},t)$ is the quantum field operator, $U(\mathbf{r})$ is the trapping potential, and $V(\mathbf{r}-\mathbf{r}')$ the inter-particle potential of the interaction between particles. Here $\mathbf{r} = (x, y, z)$.

In order to describe a bosonic field it is necessary to impose the following equal-time commutation rules

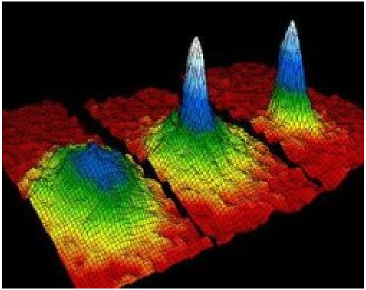
$$\left[\hat{\psi}(\mathbf{r},t), \hat{\psi}^+(\mathbf{r}',t)\right] = \delta(\mathbf{r}-\mathbf{r}') \quad (9)$$

$$\left[\hat{\psi}(\mathbf{r},t), \hat{\psi}(\mathbf{r}',t)\right] = \left[\hat{\psi}^+(\mathbf{r},t), \hat{\psi}^+(\mathbf{r}',t)\right] = 0 \quad (10)$$

[5] L. Salasnich, Quantum Physics of Light and Matter (Springer, 2017).

Bose-Einstein condensates (II)

When bosonic particles are cooled below the **critical temperature** T_c, they lose their individuality and form a "single entity" where all particles occupy the same state.



This phenomenon is known as a Bose-Einstein condensation (BEC). In 1995 for the first time BEC was achieved cooling gases of ^{87}Rb and ^{23}Na with critical temperature $T_c \simeq 100$ nanoKelvin.

Bose-Einstein condensates (III)

In the case $N \gg 1$, with N number of particles, and $T \ll T_c$, one can use the many-body coherent state $|\Psi_{cs}\rangle$, such that

$$\hat{\psi}(\mathbf{r},t)|\Psi_{cs}\rangle = \psi(\mathbf{r},t)|\Psi_{cs}\rangle \tag{11}$$

where $\psi(\mathbf{r},t)$ is the macroscopic complex wavefunction of the Bose-Einstein condensate (BEC) normalized to N, i.e.

$$\int d^3\mathbf{r}\,|\psi(\mathbf{r},t)|^2 = N \tag{12}$$

In addition, for ultracold and dilute alkali-metal atoms one can safely impose a binary contact interaction

$$V(\mathbf{r}-\mathbf{r}') = \gamma\,\delta(\mathbf{r}-\mathbf{r}') \tag{13}$$

with $\gamma = 4\pi\hbar^2 a_s/m$ where a_s is the s-wave scattering length.

Bose-Einstein condensates (IV)

In this way one obtains the following Hamiltonian:

$$\hat{H}_{\rm flat} = \int d^3r \, \left\{ \hat{\psi}^+(\mathbf{r},t) \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) \right] \hat{\psi}(\mathbf{r},t) + \frac{\gamma}{2} \hat{\psi}^+(\mathbf{r},t)^2 \hat{\psi}(\mathbf{r},t)^2 \right\} \tag{14}$$

from which, by using the coherent states, the Gross-Pitaevskii equation (GPE) is obtained

$$i\hbar\,\partial_t \psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + \gamma |\psi(\mathbf{r},t)|^2 \right] \psi(\mathbf{r},t) \tag{15}$$

This time-dependent nonlinear Schrödinger equation describes quite accurately the experiments involving pure BECs made of alkali-metal atoms.

BECs in curved spacetime (I)

As discussed by several authors[6], the GPE of a BEC in curved spacetime is given by

$$i\hbar\,\partial_t\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla_g^2 + U(\mathbf{r}) + \gamma|\psi(\mathbf{r},t)|^2\right]\psi(\mathbf{r},t) \qquad (16)$$

where ∇_g^2 is the quite complicated Laplacian operator in curved spacetime. Imposing the arrival of a GW propagating along the z axis of the BEC, one gets

$$\begin{aligned} i\hbar\,\partial_t\psi(\mathbf{r},t) &= -\frac{\hbar^2}{2m}\left[\nabla^2 + h(z,t)\left(\partial_x^2 - \partial_y^2\right)\right]\psi(\mathbf{r},t) \\ &+ \left[U(\mathbf{r}) + \gamma|\psi(\mathbf{r},t)|^2\right]\psi(\mathbf{r},t)\,, \end{aligned} \qquad (17)$$

with $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ the Laplacian operator in flat spacetime.

[6] R. Schützhold, Phys. Rev. D **98**, 105019 (2018); A. Roitberg, J. Phys.: Conf. Ser. **1730**, 012017 (2021).

BECs in curved spacetime (II)

The quantum-field-theory Hamiltonian of the BEC interacting with a gravitational wave propagating along the z-axis can be expressed as follows

$$\hat{H} = \hat{H}_{\rm flat} + \hat{H}_{\rm int} \tag{18}$$

where

$$\hat{H}_{\rm flat} = \int d^3\mathbf{r} \left\{ \hat{\psi}^{+}(\mathbf{r},t) \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) \right] \hat{\psi}(\mathbf{r},t) + \frac{\gamma}{2} \hat{\psi}^{+}(\mathbf{r},t)^2 \hat{\psi}(\mathbf{r},t)^2 \right\} \tag{19}$$

is the Hamiltonian of the bosonic quantum field in flat spacetime, while

$$\hat{H}_{\rm int} = \int \hat{\psi}^{+}(\mathbf{r},t) \left(-\frac{\hbar^2}{2m} h(z,t)(\partial_x^2 - \partial_y^2) \right) \hat{\psi}(\mathbf{r},t)\, d^3\mathbf{r} \tag{20}$$

takes into account the interaction with the GW.

Quantum fidelity (I)

At this point, we introduce the many-body coherent state $|\Psi_{cs}\rangle$ which satisfies the following eigenvalue equation

$$\hat{\psi}(\mathbf{r},t)|\Psi_{cs}\rangle = \psi_{gs}(\mathbf{r},t)|\Psi_{cs}\rangle = \sqrt{N}\,\phi_{gs}(\mathbf{r})\,e^{-\frac{i}{\hbar}\mu t}|\Psi_{cs}\rangle \tag{21}$$

where $\psi_{gs}(\mathbf{r},t)$ is normalized to N, while $\phi_{gs}(\mathbf{r})$, normalized to one, is the ground-state wavefunction of the BEC in flat spacetime with chemical potential μ. Explicitly, $\phi_{gs}(\mathbf{r})$ satisfies the stationary GPE equation in flat spacetime

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + \gamma N|\phi_{gs}(\mathbf{r})|^2\right]\phi_{gs}(\mathbf{r}) = \mu\,\phi_{gs}(\mathbf{r}) \tag{22}$$

with $\phi_{gs}(\mathbf{r})$ normalized to one, i.e.

$$\int d^3\mathbf{r}\,|\phi_{gs}(\mathbf{r})|^2 = 1 \tag{23}$$

Quantum fidelity (II)

We can now define the time evolution operator $\hat{U}_{\rm int}(t)$ associated with the interaction of the gravitational wave

$$\hat{U}_{\rm int}(t) = e^{-\frac{i}{\hbar}\int_0^t \hat{H}_{\rm int}(t')\,dt'} \tag{24}$$

where

$$|\Psi_{cs,int}\rangle = \hat{U}_{\rm int}(t)|\Psi_{cs}\rangle \tag{25}$$

is the interacting many-body quantum state at time t.
We also introduce the fidelity amplitude $\mathcal{F}(t)$, a complex number which measures how much the state remains unchanged over time:[7]

$$\mathcal{F}(t) = \langle\Psi_{cs}|\Psi_{cs,int}\rangle = \langle\Psi_{cs}|\hat{U}_{\rm int}(t)|\Psi_{cs}\rangle \tag{26}$$

Clearly, $|\mathcal{F}(t)| \in [0,1]$ where $\mathcal{F}(t) = 1$ means no change, while $\mathcal{F}(t) = 0$ signals a complete change, i.e. zero fidelity.

[7] $F(t) = |\mathcal{F}(t)|^2$ is the more familiar fidelity, a non negative real number.

Quantum fidelity (III)

By expanding the $U_{\rm int}(t)$ in a Taylor series with respect to Nh, we obtain the following result:

$$\mathcal{F}(t) = 1 - i\,N\xi(t) + \mathcal{O}(N^2 h^2) \tag{27}$$

with the N-times enhanced phase shift

$$N\xi(t) = N\frac{\hbar}{2m}\int_0^t dt' \int d^3r\, h(z,t')\phi^*_{gs}(\mathbf{r})(\partial_y^2 - \partial_x^2)\phi_{gs}(\mathbf{r}) \tag{28}$$

The proposal[8] of Schützhold is that in the future it will be possible to measure the phase shift $N\xi(t)$ by using BECs made of alkali-metal atoms.

[8] R. Schützhold, Phys. Rev. D **98**, 105019 (2018).

Phase shift for BEC in harmonic trap (I)

Here we calculate $N\xi(t)$ in the case of a BEC trapped by a harmonic potential

$$U(\mathbf{r}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \tag{29}$$

By using the technique of Feshbach resonances[9] it is now possible[10] to vary the interaction strength between the particles in a condensate, changing the scattering length a_s even to the point of making it zero. For non-interacting bosons in the harmonically trapped BEC, the ground-state wavefunction is given by

$$\phi_{gs}(\mathbf{r}) = \frac{1}{(\pi^3\sigma_x^2\sigma_y^2\sigma_z^2)^{\frac{1}{4}}} \exp\left[-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} + \frac{z^2}{2\sigma_z^2}\right)\right] \tag{30}$$

where

$$\sigma_i = \sqrt{\frac{\hbar}{m\omega_i}} \quad \text{with } i = x, y, z \tag{31}$$

[9] H. Feshbach, Ann. Phys. **5**, 357 (1958)
[10] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. **82**, 1225 (2010).

Phase shift for BEC in harmonic trap (II)

The corresponding phase shift reads

$$N\xi(t) = N\frac{\hbar}{2m}\frac{\sigma_y - \sigma_x}{\sqrt{\pi}\sigma_x\sigma_y\sigma_z}\int_0^t dt' \int_{-\infty}^{+\infty} dz \; h(z,t') \; e^{-\frac{z^2}{\sigma_z^2}} \tag{32}$$

Note that if $\sigma_x = \sigma_y$ the GW does not produce observable effects on the BEC.

By using the simplest possible expression for the GW, namely

$$h(t,z) = h_0 \cos(k_g z - \omega_g t) \tag{33}$$

we find

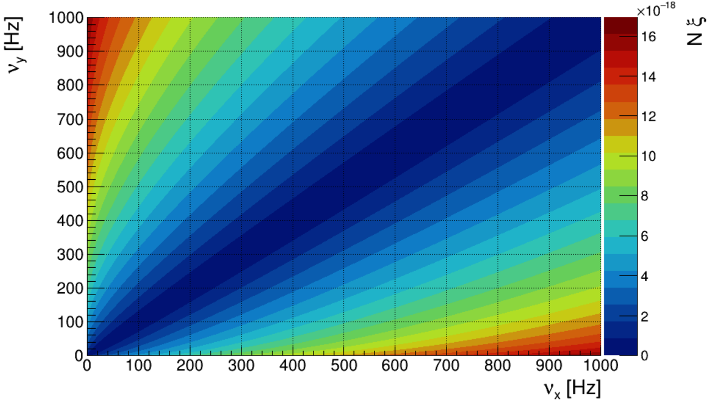
$$N\xi(t) = \frac{h_0}{2}\sqrt{\frac{\hbar}{m}}\left(\sqrt{\omega_x} - \sqrt{\omega_y}\right) \; e^{-\frac{\hbar}{4mc^2}\frac{\omega_g^2}{\omega_z}} \sin(\omega_g t) \tag{34}$$

This equation gives the phase shift, due to a GW, on a non-interacting BEC subjected to an anisotropic harmonic trapping potential. Its maximum (amplitude), given by

$$\max[N\xi(t)] = \frac{h_0}{2}\sqrt{\frac{\hbar}{m}}\left(\sqrt{\omega_x} - \sqrt{\omega_y}\right) \; e^{-\frac{\hbar}{4mc^2}\frac{\omega_g^2}{\omega_z}} \tag{35}$$

is obtained when $\sin(\omega_g t) = 1$, i.e. for instance when $\omega_g t = \pi/2$. Notice that only for $\frac{\omega_g}{\omega_z} = \mathcal{O}(10^{17})$ the damping term plays a role.

Phase shift for BEC in harmonic trap (III)



Phase shift $N\xi(t)$ at $\omega_g t = \pi/2$ as a function of the confinement frequencies $\nu_x = \omega_x/(2\pi)$ and $\nu_y = \omega_y/(2\pi)$ in the xy plane. BEC of non-interacting ^{87}Rb atoms, with fixed values of $\omega_z = 2\pi \times 150\,\mathrm{Hz}$, $h_0 = 10^{-20}$ and $N = 10^7$.

Phase shift for BEC in harmonic trap (IV)

It is possible to make use a Gaussian variational method[11] to evaluate also the case of an interacting BEC. The Gaussian trial function is the same of Eq. (30) but the σ_i are variational parameters. In the strongly interacting regime, the σ_i^* that minimize the ground-state energy are

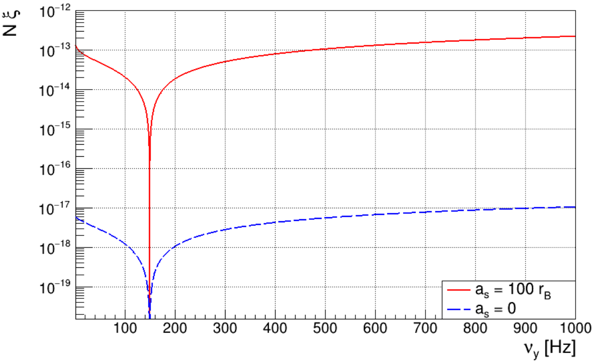
$$\sigma_x^* = \left(\frac{\Gamma\hbar^{\frac{3}{5}}\omega_y\omega_z}{m\omega_x^4}\right)^{\frac{1}{5}} \quad \sigma_y^* = \left(\frac{\Gamma\hbar^{\frac{3}{5}}\omega_z\omega_x}{m\omega_y^4}\right)^{\frac{1}{5}} \quad \sigma_z^* = \left(\frac{\Gamma\hbar^{\frac{3}{5}}\omega_x\omega_y}{m\omega_z^4}\right)^{\frac{1}{5}} \tag{36}$$

where $\Gamma = \gamma \frac{N}{(2\pi)^{\frac{3}{2}}}$. Then the procedure for calculating $N\xi(t)$ leads to

$$N\xi(t) = N\frac{h_0}{2}\left(\frac{\hbar^{\frac{11}{5}}}{m^2\sqrt{\Gamma\omega_z}}\right)^{\frac{2}{5}} e^{-\frac{\omega_g^2}{4c^2}\left(\frac{\Gamma\hbar^{\frac{3}{5}}\omega_x\omega_y}{m\omega_z^4}\right)^{\frac{2}{5}}} \left(\left(\frac{\omega_x^4}{\omega_y}\right)^{\frac{1}{5}} - \left(\frac{\omega_y^4}{\omega_x}\right)^{\frac{1}{5}}\right)\sin(\omega_g t) \tag{37}$$

[11] L. Salasnich, Int. J. Mod. Phys B **14**, 1 (2000).

Phase shift for BEC in harmonic trap (V)



Phase shift $N\xi(t)$ at $\omega_g t = \pi/2$ for a BEC of interacting ^{87}Rb atoms, plotted as a function of the confinement frequency $\nu_y = \omega_y/(2\pi)$ along the y-axis. Fixed parameters: $\nu_z = 150\,\mathrm{Hz}$, $\nu_x = 200\,\mathrm{Hz}$, $h_0 = 10^{-20}$, $a_S = 100 a_B$, and $N = 10^7$. a_B is the Bohr radius.

Conclusions

- We have discussed the interaction Hamiltonian between a gravitational wave and a Bose-Einstein condensate.
- We have computed the fidelity amplitude at first order with respect to Nh, where N is the number of atoms in the Bose-Einstein condensate and h is a scalar component of gravitational wave.
- We have shown an enhancement for the phase shift of the fidelity amplitude that is proportional to the number N of condensed atoms.
- We have explicitly evaluated the magnitude of the phase shift in the case of Bose-Einstein condensates confined by an anisotropic harmonic potential.
- The experimental detection of the phase shift is still a **puzzling problem** under discussion.
- Take-home message: at fixed number N of atoms, tuning the s-wave scattering length a_s of the inter-atomic interaction one can increase of several order of magnitude the many-body phase $N\xi(t)$ which signals the arrival of a gravitational wave.

Acknowledgements

Thank you for your attention!