Vortices and Antivortices in
Two-Dimensional Ultracold Fermi Gases

Luca Salasnich

Dipartimento di Fisica e Astronomia "Galileo Galilei" and CNISM, Università di Padova
INO-CNR, Research Unit of Sesto Fiorentino, Consiglio Nazionale delle Ricerche

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Giacomo Bighin and Flavio Toigo

Summary

- BCS-BEC crossover in 2D
- Quantum fluctuations in 2D
- Vortices and antivortices in 2D
- Conclusions

BCS-BEC crossover in 2D (I)

In 2004 the 3D BCS-BEC crossover has been observed with **ultracold gases made of two-component fermionic ^{40}K or ^{6}Li alkali-metal atoms**.[1]

BCS   BEC

Cooper pairs   strongly interacting pairs   diatomic molecules
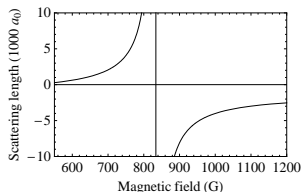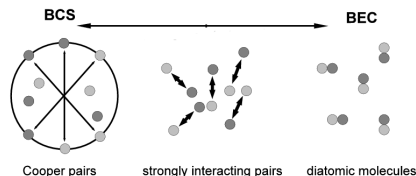
This crossover is obtained by using a Fano-Feshbach resonance to change the 3D s-wave scattering length a_s of the inter-atomic potential

$$a_s = a_{bg} \left(1 + \frac{\Delta_B}{B - B_0} \right) , \qquad (1)$$

where B is the external magnetic field.

[1]C.A. Regal et al., PRL **92**, 040403 (2004); M.W. Zwierlein et al., PRL **92**, 120403 (2004); J. Kinast et al., PRL **92**, 150402 (2004).

Recently also the 2D BEC-BEC crossover has been achieved experimentally[2] with a **Fermi gas of two-component ^{6}Li atoms**. In 2D attractive fermions <u>always</u> form biatomic molecules with bound-state energy

$$\epsilon_B \simeq \frac{\hbar^2}{m a_s{}^2} , \tag{2}$$

where a_s is the 2D s-wave scattering length, which is experimentally tuned by a Fano-Feshbach resonance.

The fermionic single-particle spectrum is given by

$$E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta^2} , \tag{3}$$

where Δ is the energy gap and μ is the chemical potential: $\mu > 0$ corresponds to the BCS regime while $\mu < 0$ corresponds to the BEC regime. Moreover, in the deep BEC regime $\mu \to -\epsilon_B/2$.

[2]V. Makhalov et al. PRL **112**, 045301 (2014); M.G. Ries et al., PRL **114**, 230401 (2015); I. Boettcher et al., PRL **116**, 045303 (2016).

BCS-BEC crossover in 2D (III)

To study the 2D BCS-BEC crossover we adopt the formalism of functional integration[3]. The partition function $\mathcal{Z}$ of the uniform system with fermionic fields $\psi_s(\mathbf{r}, \tau)$ at temperature T, in a D-dimensional volume L^D, and with chemical potential μ reads

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \exp\left\{-\frac{S}{\hbar}\right\}, \tag{4}$$

where ($\beta \equiv 1/(k_B T)$ with k_B Boltzmann's constant)

$$S = \int_0^{\hbar\beta} d\tau \int_{L^D} d^D\mathbf{r} \, \mathcal{L} \tag{5}$$

is the Euclidean action functional with Lagrangian density

$$\mathcal{L} = \bar{\psi}_s \left[\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu\right]\psi_s + \mathbf{g}\,\bar{\psi}_\uparrow\,\bar{\psi}_\downarrow\,\psi_\downarrow\,\psi_\uparrow \tag{6}$$

where $\mathbf{g}$ is the attractive strength ($\mathbf{g} < 0$) of the s-wave coupling.

[3] N. Nagaosa, Quantum Field Theory in Condensed Matter Physics (Springer, 1999)

Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathcal{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(\mathbf{r}, \tau)$. In this way the effective Euclidean Lagrangian density reads

$$\mathcal{L}_e = \bar{\psi}_s \left[\hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + \bar{\Delta}\, \psi_\downarrow\, \psi_\uparrow + \Delta \bar{\psi}_\uparrow\, \bar{\psi}_\downarrow - \frac{|\Delta|^2}{\mathbf{g}} \; . \quad (7)$$

We investigate the effect of fluctuations of the gap field $\Delta(\mathbf{r}, t)$ around its mean-field value Δ_0 which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r}, \tau) = \Delta_0 + \eta(\mathbf{r}, \tau) \; , \quad (8)$$

where $\eta(\mathbf{r}, \tau)$ is the complex field which describes pairing fluctuations.

In particular, we are interested in the grand potential Ω, given by

$$\Omega = -\frac{1}{\beta} \ln\left(\mathcal{Z}\right) \simeq -\frac{1}{\beta} \ln\left(\mathcal{Z}_{mf}\,\mathcal{Z}_g\right) = \Omega_{mf} + \Omega_B \ , \qquad (9)$$

where

$$\mathcal{Z}_{mf} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \exp\left\{-\frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar}\right\} \qquad (10)$$

is the mean-field partition function and

$$\mathcal{Z}_g = \int \mathcal{D}[\psi_s, \bar{\psi}_s]\,\mathcal{D}[\eta, \bar{\eta}] \, \exp\left\{-\frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar}\right\} \qquad (11)$$

is the partition function of Gaussian pairing fluctuations.

One finds that in the gas of paired fermions there are two kinds of elementary excitations: fermionic single-particle excitations with energy

$$E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2} \,, \qquad (12)$$

where Δ_0 is the pairing gap, and bosonic collective excitations with energy (at low-momenta and in the BEC regime)

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m} \left(\lambda \, \frac{\hbar^2 q^2}{2m} + 2 \, m \, c_s^2\right)} \,, \qquad (13)$$

where λ is the first correction to the familiar phonon dispersion $E_{col}(q) \simeq c_s \hbar q$ and c_s is the sound velocity. Notice that both λ and c_s depend on the chemical potential μ.

Quantum fluctuations in 2D (II)

In the analysis of the **two-dimensional attractive Fermi gas** one must remember that, contrary to the 3D case, 2D realistic interatomic attractive potentials have always a bound state. In particular[4], the binding energy $\epsilon_B > 0$ of two fermions can be written in terms of the positive 2D fermionic scattering length a_s as

$$\epsilon_B = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{m a_s^2} \, , \tag{14}$$

where $\gamma = 0.577...$ is the Euler-Mascheroni constant. Moreover, the attractive (negative) interaction strength **g** of s-wave pairing is related to the binding energy $\epsilon_B > 0$ of a fermion pair in vacuum by the expression[5]

$$-\frac{1}{\mathbf{g}} = \frac{1}{2L^2} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 k^2}{2m} + \frac{1}{2}\epsilon_B} \, . \tag{15}$$

[4] C. Mora and Y. Castin, 2003, PRA **67**, 053615.
[5] M. Randeria, J-M. Duan, and L-Y. Shieh, PRL **62**, 981 (1989).

Quantum fluctuations in 2D (III)

In the **2D BCS-BEC crossover**, at zero temperature ($T = 0$) the mean-field grand potential Ω_{mf} can be written as[6] ($\epsilon_B > 0$)

$$\Omega_{mf} = -\frac{mL^2}{2\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2 \ . \tag{16}$$

Using

$$n = -\frac{1}{L^2}\frac{\partial\Omega_{mf}}{\partial\mu} \tag{17}$$

one immediately finds the chemical potential μ as a function of the number density $n = N/L^2$, i.e.

$$\mu = \frac{\pi\hbar^2}{m}n - \frac{1}{2}\epsilon_B \ . \tag{18}$$

In the BCS regime, where $\epsilon_B \ll \epsilon_F$ with $\epsilon_F = \pi\hbar^2 n/m$, one finds $\mu \simeq \epsilon_F > 0$ while in the BEC regime, where $\epsilon_B \gg \epsilon_F$ one has $\mu \simeq -\epsilon_B/2 < 0$.

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[6]M. Randeria, J-M. Duan, and L-Y. Shieh, PRL **62**, 981 (1989).

In the deep BEC regime of the **2D BCS-BEC crossover**, where the chemical potential μ becomes negative, performing regularization of zero-point fluctuations we have recently found[7] that the zero-temperature grand potential (including bosonic excitations) is

$$\Omega = -\frac{mL^2}{64\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2 \; \ln\left(\frac{\epsilon_B}{2(\mu + \frac{1}{2}\epsilon_B)}\right) \; . \qquad (19)$$

This is exactly Popov's equation of state of 2D Bose gas with chemical potential $\mu_B = 2(\mu + \epsilon_B/2)$ and mass $m_B = 2m$. In this way we have identified the two-dimensional scattering length a_B of composite bosons as
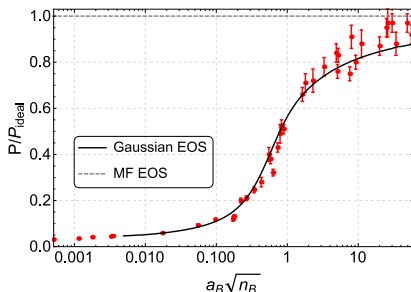
$$\boxed{a_B = \frac{1}{2^{1/2}e^{1/4}} \; a_s \; .} \qquad (20)$$

The value $a_B/a_s = 1/(2^{1/2}e^{1/4}) \simeq 0.551$ is in full agreement with $a_B/a_s = 0.55(4)$ obtained by Monte Carlo calculations[8].

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[7]LS and F. Toigo, PRA **91**, 011604(R) (2015).
[8]G. Bertaina and S. Giorgini, PRL **106**, 110403 (2011).

Scaled pressure $P/(2P_{ideal})$ as a function of the bosonic gas parameter $a_B n_B^{1/2}$, where $P_{ideal} = 2\pi\hbar^2 n_B^2/m_B$ is the pressure of an ideal 2D gas with $m_B = 2m$ the mass of each bosonic particle (made of two fermions with mass m), a_B is the s-wave scattering length of bosons, and $n_B = n/2$ is the bosonic 2D density (with n the fermionic density). Filled squares with error bars are experimental data of Makhalov et al. [9]. Solid line is obtained with our renormalized Gaussian theory[10]

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[9]V. Makhalov et al. PRL **112**, 045301 (2014)
[10]G. Bighin and LS, PRB **93**, 014519 (2016).

Vortices and antivortices in 2D (I)

The low-energy Hamiltonian of a 2D fermionic superfluid can be recast as that of an effective continuous 2D XY model

$$H = \frac{J(T)}{2} \int d^2\mathbf{r} \left(\nabla\theta(\mathbf{r})\right)^2 , \qquad (21)$$

where $\theta(\mathbf{r})$ is the phase angle of the pairing field $\Delta(\mathbf{r}) = |\Delta(\mathbf{r})|e^{i\theta(\mathbf{r})}$ and

$$J(T) = \frac{\hbar^2}{4m} n_s(T) \qquad (22)$$

is the phase stiffness and $n_s(T)$ is the superfluid density. The compactness of the phase angle $\theta(\mathbf{r})$ implies that

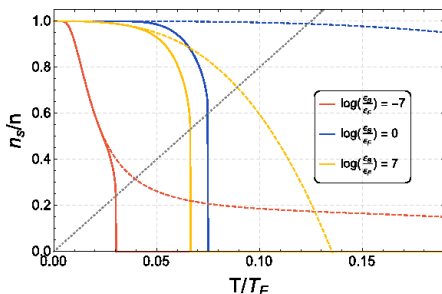$$\oint \nabla\theta(\mathbf{r}) \cdot d\mathbf{r} = 2\pi q , \qquad (23)$$

where q is the integer number associated to quantum vortices ($q > 0$) and antivortices ($q < 0$), which renormalize[11] $n_s(T)$ and $J(T)$.

[11] J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973).

Vortices and antivortices in 2D (II)

In our approach the (bare) superfluid density is given by

$$n_s(T) = n - \beta \int \frac{\mathrm{d}^2 k}{(2\pi)^2} k^2 \frac{e^{\beta E_{sp}(k)}}{(e^{\beta E_{sp}(k)} + 1)^2} - \frac{\beta}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} q^2 \frac{e^{\beta E_{col}(q)}}{(e^{\beta E_{col}(q)} - 1)^2} \tag{24}$$

The Berezinskii-Kosterlitz-Thouless critical temperature T_{BKT} is determined by the jump of the renormalized superfluid density $n_{s,r}(T)$, which derived from **Kosterlitz's renormalization-group equations**.[12]

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[12]J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973).
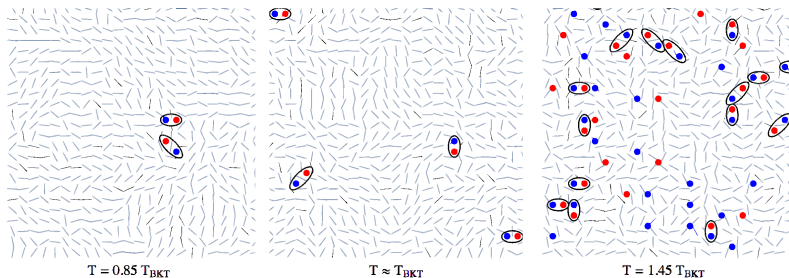
Vortices and antivortices in 2D (III)
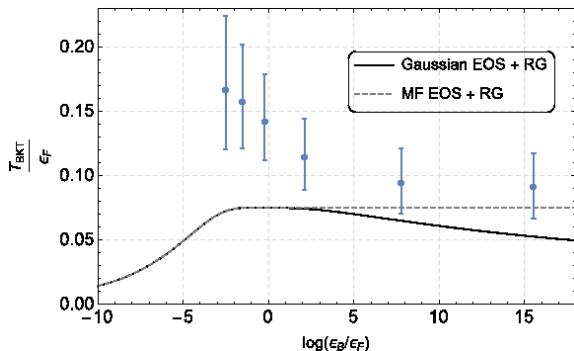
Theoretical predictions for the Berezinskii-Kosterlitz-Thouless critical temperature T_{BKT} (at which vortex-antivortex pairs unbind) compared[13] to recent experimental observation[14] (circles with error bars).

[13]G. Bighin and LS, PRB **93**, 014519 (2016); G. Bighin and LS, in preparation.
[14]P.A. Murthy et al., PRL **115**, 010401 (2015).

T = 0.85 T_{BKT} T ≈ T_{BKT} T = 1.45 T_{BKT}

Monte Carlo simulation of the **vortex-antivortex unbinding** transition at T_{BKT}.[15] Arrows denote the local gradient of the phase field, i.e. $\nabla\theta(\mathbf{r})$, while the red and blue dots correspond to vortices with ± 1 topological charge, i.e. vortices and antivortices.

A black ellipse is drawn around bound vortices, to guide the eye in visualizing the vortex unbinding transition. The bound state energy is $\log(\epsilon_B/\epsilon_F) = 5$.

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[15] Discrete XY model in a 64×64 lattice with Wolff algorithm. G. Bighin and LS, in preparation.

Vortices and antivortices in 2D (V)
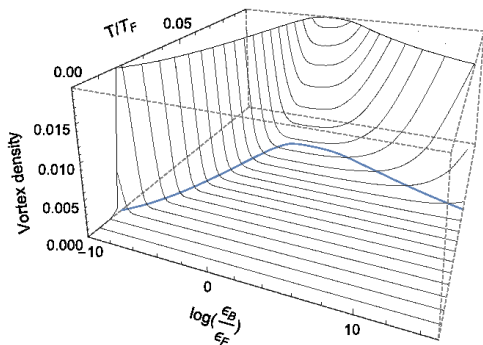
Vortex density as a function of the fermion-fermion interaction strength and of the temperature, as calculated from Monte Carlo simulations.[16] The blue line denotes the BKT critical temperature.

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[16]G. Bighin and LS, in preparation.

Conclusions

- The regularization of zero-point energy[17] gives remarkable beyond-mean-field effects for composite bosons in the 2D BCS-BEC crossover at zero temperature:
  - logarithmic behavior of the equation of state
  - Bose-Bose scattering length a_B vs Fermi-Fermi scattering length a_s
  - speed of first sound (and also second sound)

- Also at finite temperature beyond-mean-field effects, with the inclusion of quantized vortices and antivortices, become relevant in the strong-coupling regime of 2D BCS-BEC crossover:
  - superfluid density n_s
  - critical temperature T_{BKT}
  - proliferation of quantized vortices

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[17]For a very recent comprehensive review see:
LS and F. Toigo, Zero-Point Energy of Ultracold Atoms,
Physics Reports 640, 1 (2016).

Thank you for your attention!