Amplitude, phase, and topological fluctuations in two-dimensional superconductors

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Summary

- 2D superfluids and BKT
- 2D Ginzburg-Landau functional
- Gaussian fluctuations of the order parameter
- Topological phase fluctuations of the order parameter
- Critical temperatures
- Conclusions

2D superfluids and BKT

In two-dimensional (2D) superconductors or superfluids, the vortex excitations lead to a topological phase transition without spontaneous symmetry breaking, which is referred to as the Berezinskii-Kosterlitz-Thouless (BKT) phase transition[1].

Clear signatures of BKT transitions were first observed in a thin $^4\mathrm{He}$ film[2]. Later, it was experimentally observed also in thin and disordered superconducting films[3] and ultracold 2D atomic gases[4].

[1] V.L. Berezinskii, Sov. Phys. JETP **32**, 493 (1971); J. M. Kosterlitz and D. J. Thouless, J. Phys. C: Solid State Phys. **6**, 1181 (1973); D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. **39**, 1201 (1977).

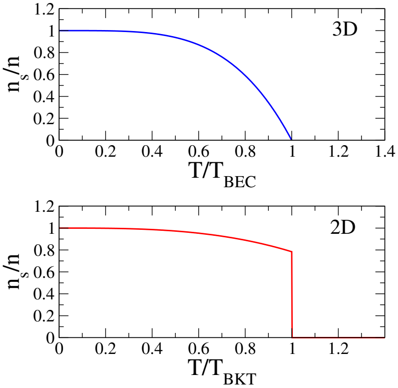
[2] D. J. Bishop and J. D. Reppy, Phys. Rev. Lett. **40**, 1727 (1978).

[3] M. Mondal *et al.*, Phys. Rev. Lett. **107**, 217003 (2011).

[4] P. Christodoulou *el al.*, Nature **594**, 191 (2021).

2D superfluids and BKT

The main prediction of the Kosterlitz-Thouless transition is that, contrary to the 3D case, in 2D superfluids the superfluid fraction n_s/n jumps to zero above a critical temperature.

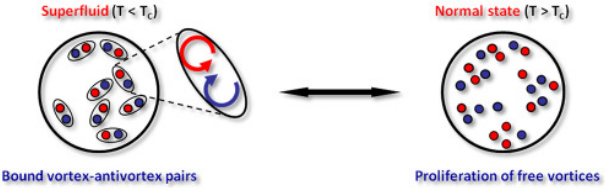


For 3D superfluids the transition to the normal state is a BEC phase transition, while in 2D superfluids the transition to the normal state is something different: a topological phase transition.

2D superfluids and BKT

The analysis of **Kosterlitz** and **Thouless** applied to 2D superfluids shows that:

- As the temperature T increases vortices start to appear in vortex-antivortex pairs.
- The pairs are bound at low temperature until at the Berezinskii-Kosterlitz-Thouless critical temperature $T_c = T_{BKT}$ an unbinding transition occurs above which a proliferation of free vortices and antivortices is predicted.
- The superfluid density $n_s(T)$ is renormalized by the presence of vortex-antivortex pairs.
- The renormalized superfluid density $n_{s,R}(T)$ decreases by increasing the temperature T and jumps to zero at $T_c = T_{BKT}$.



2D superfluids and BKT

We have seen that the renormalized superfluid density $n_{s,R}(T)$ jumps to zero at a critical temperature T_{BKT}.
Moreover, one can prove the Nelson-Kosterlitz condition[5]

$$k_B T_{BKT} = \frac{\hbar^2 \pi}{8m} n_{s,R}(T_{BKT}^-) \; . \qquad (1)$$

Often the following Nelson-Kosterlitz criterion is adopted

$$k_B T_{BKT} = \frac{\hbar^2 \pi}{8m} n_s(T_{BKT}) \; , \qquad (2)$$

with $n_s(T)$ **instead of** $n_{s,R}(T)$. In this way one gets an approximated T_{BKT} without the effort of calculating the renormalized superfluid density $n_{s,R}(T)$.

[5] D.R. Nelson and J.M. Kosterlitz, Phys Rev. Lett. **39**, 1201 (1977).

2D Ginzburg-Landau functional

Here we investigate 2D superconductors by using the Ginzburg-Landau theory[6]. In this framework, we analyze the effects of the thermal amplitude fluctuations and phase fluctuations, which result in the BKT transition.

Close to the critical temperature the free energy of a single-band superconducting material can be written as

$$F = F_{\mathrm{n}} + F_{\mathrm{s}} \ , \tag{3}$$

where F_{n} is the contribution due to the normal component and F_{s} is the contribution due to the emergence of a superconducting order parameter $\psi(\mathbf{r})$ below the critical temperature.

[6] V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950)

2D Ginzburg-Landau functional

Within the Ginzburg-Landau approach, for a two-dimensional system of area L^2, the super component $F_{\rm s}$ is given by the following Ginzburg-Landau functional

$$F_{\rm s} = \int_{L^2} d^2\mathbf{r} \left\{ a(T)\,|\psi(\mathbf{r})|^2 + \frac{b}{2}\,|\psi(\mathbf{r})|^4 + \gamma\,|\boldsymbol{\nabla}\psi(\mathbf{r})|^2 \right\} , \qquad (4)$$

where

$$a(T) = \alpha\, k_{\rm B}\,(T - T_{\rm c0}) \qquad (5)$$

is a parameter which depends on the temperature T and becomes zero at the mean-field (MF) critical temperature $T_{\rm c0}$, while $b > 0$ and $\gamma > 0$ are temperature-independent parameters with $k_{\rm B}$ being the Boltzmann constant.

The energy functional (4) and the values of the parameters α, b and γ can be deduced from the microscopic Bardeen-Cooper-Schrieffer (BCS) theory[7], as shown by Gorkov[8].

[7] J. Bardeen, L.N. Cooper, and J.R. Schrieffer, Phys. Rev. **106**, 162 (1957).
[8] L. P. Gorkov, Sov. Phys. JETP **36**, 1364 (1959).

2D Ginzburg-Landau functional

The partition function $\mathcal{Z}$ of the system is given by

$$\mathcal{Z} = e^{-\beta F_{\rm n}} \; \mathcal{Z}_{\rm s} \;, \tag{6}$$

where

$$\mathcal{Z}_{\rm s} = \int \mathcal{D}[\psi(\mathbf{r})] \, e^{-\beta F_{\rm s}[\psi(\mathbf{r})]} \tag{7}$$

is the partition function of the superconducting component with $\beta = 1/(k_{\rm B} T)$. The thermal average of an observable $\mathcal{O}$ that is a functional of $\psi(\mathbf{r})$ reads

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}_{\rm s}} \int \mathcal{D}[\psi(\mathbf{r})] \, \mathcal{O}[\psi(\mathbf{r})] \, e^{-\beta F_{\rm s}[\psi(\mathbf{r})]} \;. \tag{8}$$

2D Ginzburg-Landau functional

By assuming a real and uniform order parameter, i.e.

$$\psi(\mathbf{r}) = \psi_0 \; , \tag{9}$$

the energy functional (4) with Eqs. (5) and (9) becomes

$$\frac{F_{s0}[\psi_0]}{L^2} = a(T)\, \psi_0^2 + \frac{b}{2}\, \psi_0^4 \; . \tag{10}$$

Minimizing F_{s0} with respect to ψ_0, one immediately finds

$$a(T)\, \psi_0 + b\, \psi_0^3 = 0 \; , \tag{11}$$

2D Ginzburg-Landau functional

and consequently

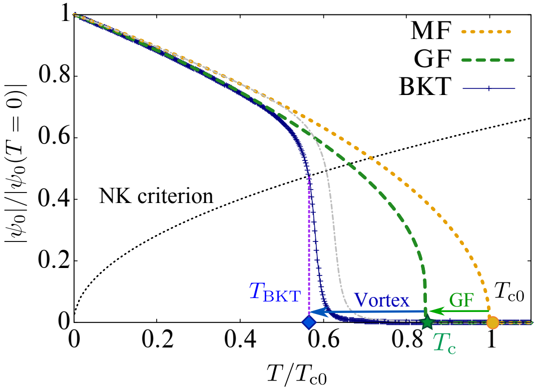
$$\psi_0 = \begin{cases} 0 & \text{for } T \geq T_{c0}, \\ \sqrt{-\dfrac{a(T)}{b}} & \text{for } T < T_{c0}. \end{cases} \tag{12}$$

The uniform order parameter ψ_0 becomes different from zero only below the MF critical temperature T_{c0}, where, by definition, T_{c0} is such that

$$a(T_{c0}) = 0 \; . \tag{13}$$

The temperature dependence of the MF order parameter (12) is shown in the next figure by a dotted yellow curve.

2D Ginzburg-Landau functional



Order parameters as a function of the temperature T. The order parameter ψ_0 obtained at the mean-field (MF) level is illustrated with a dotted yellow curve. The inclusion of Gaussian fluctuations (GF) is illustrated with a dashed green curve. The other curves takes also into account phase fuctuations, i.e. the inclusion the vortex excitations responsible for the BKT transition. **K.Furutani, G. Midei, A. Perali, and LS, Phys. Rev. B 110, 134501 (2024)**.

Gaussian fluctuations of the order parameter

Let us see how to include Gaussian fluctuations of the order parameter. Extremizing the Ginzburg-Landau functional (4) with respect to $\psi^*(\mathbf{r})$ one gets the Euler-Lagrange equation

$$a(T)\,\psi + b\,|\psi|^2\psi - \gamma\,\nabla^2\psi = 0\ . \tag{14}$$

We write the space-dependent order parameter $\psi(\mathbf{r})$ in the following way

$$\psi(\mathbf{r}) = \psi_0^{(\mathrm{GF})} + \eta(\mathbf{r})\ , \tag{15}$$

where $\eta(\mathbf{r})$ represents a small fluctuation with respect to the real and uniform configuration $\psi_0^{(\mathrm{GF})}$ with the condition

$$\langle\eta\rangle = \langle\eta^*\rangle = 0\ , \tag{16}$$

where $\langle\cdot\rangle$ is the thermal average.

Gaussian fluctuations of the order parameter

Inserting Eq. (15) into Eq. (14), and after thermal averaging we obtain

$$\left[a(T) + 2b\,\langle|\eta|^2\rangle\right]\,\psi_0^{(\mathrm{GF})} + b\,(\psi_0^{(\mathrm{GF})})^3 = 0\,, \qquad (17)$$

and consequently

$$\psi_0^{(\mathrm{GF})} = \begin{cases} 0 & \text{for } T \geq T_c, \\ \sqrt{-\dfrac{a(T) + 2b\langle|\eta|^2\rangle}{b}} & \text{for } T < T_c. \end{cases} \qquad (18)$$

In this case, the uniform order parameter $\psi_0^{(\mathrm{GF})}$ becomes different from zero only below the beyond-MF critical temperature T_{c} determined by

$$a(T_{\mathrm{c}}) + 2b\,\langle|\eta|^2\rangle_{\mathrm{c}} = 0\,, \qquad (19)$$

with $\langle|\eta|^2\rangle_{\mathrm{c}} \equiv \langle|\eta|^2\rangle_{T\to T_{\mathrm{c}}^+}$.

Gaussian fluctuations of the order parameter

Under the assumption of small fluctuations (Gaussian fluctuations), the fluctuating field $\eta(\mathbf{r})$ satisfies the following equation

$$a(T)\,\eta + 2b\,(\psi_0^{(\mathrm{GF})})^2\eta + 2b\,\langle|\eta|^2\rangle\eta + b\,(\psi_0^{(\mathrm{GF})})^2\eta^* - \gamma\nabla^2\eta = 0\,. \quad (20)$$

Moreover, one finds[9] that the thermal average $\langle|\eta|^2\rangle$ reads

$$\begin{aligned} \langle|\eta|^2\rangle &= \frac{1}{2\beta L^2}\sum_{\mathbf{k}}\left[\frac{1}{a(T)+2b\langle|\eta|^2\rangle+3b(\psi_0^{(\mathrm{GF})})^2+\gamma k^2}\right. \\ &+ \left.\frac{1}{a(T)+2b\langle|\eta|^2\rangle+b(\psi_0^{(\mathrm{GF})})^2+\gamma k^2}\right]. \end{aligned} \quad (21)$$

having expanded the field $\eta(\mathbf{r})$ inside a 2D square box of size L as follows

$$\eta(\mathbf{r}) = \frac{1}{L}\sum_{\mathbf{k}}\tilde{\eta}_{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{r}}\,. \quad (22)$$

[9] K. Furutani, G. Midei, A. Perali, and LS, Phys. Rev. B **110**, 134501 (2024).

Topological phase fluctuations of the order parameter

Usually[10] the crucial role of phase fluctuations, i.e. the effect of quantized vortices, is taken into account writing the space-dependent order parameter $\psi(\mathbf{r})$ in this way

$$\psi(\mathbf{r}) = \psi_0 \, e^{i\theta(\mathbf{r})} \;, \tag{23}$$

where $\theta(\mathbf{r})$ is the Nambu-Goldstone phase field and ψ_0 the MF order parameter.
Here we consider an improved ansatz, which includes both Gaussian and phase fluctuations, namely

$$\psi(\mathbf{r}) = \left[\psi_0^{(BKT)} + \eta(\mathbf{r})\right] e^{i\theta(\mathbf{r})} \;. \tag{24}$$

[10] A. Larkin and A. Varlamov, Theory of fluctuations in superconductors (Clarendon Press, Oxford, 2007).

Topological phase fluctuations of the order parameter

The compactness of the phase angle field $\theta(\mathbf{r})$ implies

$$\oint_{\mathcal{C}} \nabla\theta(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 2\pi q \tag{25}$$

for any closed contour $\mathcal{C}$. Here, $q = 0, \pm 1, \pm 2, ...$ is the integer topological charge associated with the corresponding quantized vortex (positive q) or antivortex (negative q). Thus, the field $\theta(\mathbf{r})$ can be written as

$$\theta(\mathbf{r}) = \theta_0(\mathbf{r}) + \theta_{\mathrm{vor}}(\mathbf{r}) \ , \tag{26}$$

where $\theta_0(\mathbf{r})$ is vortex free and only $\theta_{\mathrm{vor}}(\mathbf{r})$ contains nonzero topological charges. Notice that

$$\eta(\mathbf{r}) = |\eta(\mathbf{r})| \, e^{i\phi(\mathbf{r})} \tag{27}$$

with $\phi(\mathbf{r})$ a regular phase with zero winding number, such as $\theta_0(\mathbf{r})$, and $|\eta(\mathbf{r})|$ the amplitude fluctuation (Higgs mode).

Topological phase fluctuations of the order parameter

Inserting the improved ansatz Eq. (24) into the Ginzburg-Landau functional, one obtains

$$F_{\mathrm{s}} = F_{\mathrm{s}0}[\psi_0^{(\mathrm{BKT})}] + F_\eta + F_\theta \ , \tag{28}$$

where the only-phase functional reads

$$F_\theta = \frac{J_0(T)}{2} \int_{L^2} d^2\mathbf{r} \, (\nabla\tilde{\theta}(\mathbf{r}))^2 \tag{29}$$

with bare phase stiffness

$$J_0(T) = 2\gamma(\psi_0^{(BKT)})^2 = 2\gamma \frac{\Big(a(T) + 2b\langle|\eta|^2\rangle + \langle(\nabla\tilde{\theta})^2\rangle\Big)^2}{b^2} \tag{30}$$

and

$$\tilde{\theta}(\mathbf{r}) = \theta(\mathbf{r}) + \frac{\langle|\eta|^2\rangle}{(\psi_0^{(BKT)})^2}\phi(\mathbf{r}). \tag{31}$$

Topological phase fluctuations of the order parameter

The only-phase functional can be re-written as

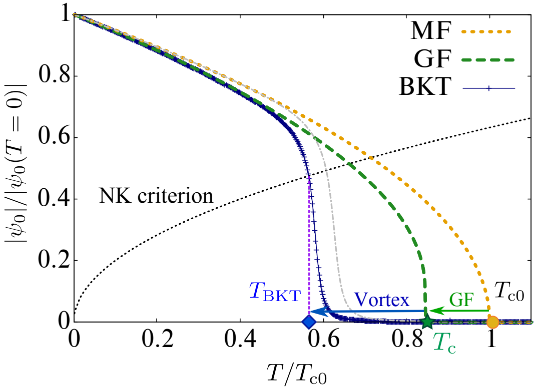
$$F_\theta = \frac{J_R(T)}{2} \int_{L^2} d^2\mathbf{r} \, (\nabla \tilde{\theta}_0(\mathbf{r}))^2 \tag{32}$$

with renormalized phase stiffness $J_R(T)$ and the vortex-free angle $\tilde{\theta}_0(\mathbf{r})$. It is important to stress that the renormalized phase stiffness $J_R(T)$ is obtained from the bare one $J_0(T)$ by adopting the Kosterlitz-Thouless renormalization group equations.

Having the $J_R(T)$ one then derives the renormalized BKT order parameter $\psi_R^{(BKT)}$ as

$$\psi_R^{(BKT)} = \sqrt{\frac{J_R(T)}{2\gamma}} \, . \tag{33}$$

Topological phase fluctuations of the order parameter



Order parameters as a function of the temperature T. The order parameter ψ_0 obtained at the mean-field (MF) level is illustrated with a dotted yellow curve. The inclusion of Gaussian fluctuations (GF) is illustrated with a dashed green curve. The other curves takes also into account phase fuctuations, i.e. the inclusion the vortex excitations responsible for the BKT transition. **K.Furutani, G. Midei, A. Perali, and LS, Phys. Rev. B 110, 134501 (2024)**.

Critical temperatures

To analyze the phase fluctuations of 2D superconductors it is useful to introduce the Ginzburg-Levaniuk number

$$Gi = \frac{b}{4\pi\alpha\gamma} = \frac{\pi T_{c0}}{2T_F} \tag{34}$$

with T_F the Fermi temperature of a 2D ideal electron gas. Our formalism is reliable under the assumption that $Gi \ll 1$.
We find the simple analytical formula

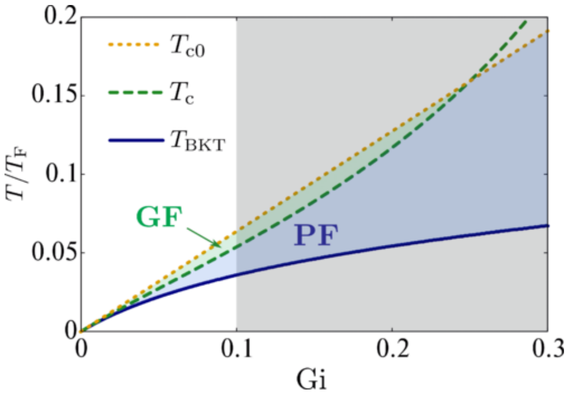
$$\frac{T_{c0} - T_c}{T_c} = 2Gi \ln\left(\frac{1}{4Gi}\right) \tag{35}$$

relating the MF critical temperature T_{c0} to T_c, which takes into account Gaussian fluctuations.
The BKT critical temperature T_{BKT}, which includes both Gaussian and phase fluctuations, must be instead calculated numerically. However, we found the remarkable formula

$$k_B T_{BKT} = 0.948\, J_0(T_{BKT}) \tag{36}$$

which depends only on the knowledge of the bare phase stiffness.

Critical temperatures



Dependence of the critical temperatures T_{c0}, T_c, and T_{BKT} in the unit of T_F as a function of Gi. The shaded region of $T_c < T < T_{c0}$ is governed by the Gaussian thermal fluctuations (GF), while the phase fluctuations (PF) associated with vortex excitations are dominant in the region of $T_{BKT} < T < T_c$.

Conclusions

- Our analysis based on the Ginzburg-Landau functional reveals that the mean-field critical temperature T_{c0} is reduced by Gaussian fluctuations to T_c.
- We found that the critical temperature T_c is further reduced to T_{BKT} by vortex excitations responsible for the BKT transition.
- These results are obtained incorporating the effects of both amplitude fluctuations and phase fluctuations in the uniform order parameter of the Ginzburg-Landau model.
- Based on our approach of cascade of fluctuations, one can also obtain (see K. Furutani, G. Midei, A. Perali, and L. Salasnich, Phys. Rev. B **110**, 134501 (2024)) the H-T phase diagram of type-II superconductors and the critical behavior of the heat capacity.

Thank you for your attention!

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