Self-bound droplets in bosonic atomic gases

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International Workshop on Atomic Physics, Max Planck Institute for the Physics of Complex Systems Dresden, November 24, 2025

Collaboration with G.F. Bertacco, A. Cappellaro, and T. Macri



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- Beyond-mean-field effective action
- Gaussian ansatz and soliton-droplet crossover
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- Spin-dipole mode
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Self-bound Bose-Bose droplets

- In 2015 Dmitry Petrov suggested theoretically¹ the existence of self-bound quantum droplets in an attractive Bose-Bose mixture, where the collapse is suppressed by a beyond-mean-field term.
- Experiments with two internal states of ³⁹K atoms in a 3D configuration, performed both at Barcelona² and Florence³, confirmed these theoretical predictions.
- In another experiment at Barcelona⁴ self-bound states of the two-component BEC were studied in a tight optical waveguide (quasi-1D confinement): a smooth crossover interpolating between bright soliton and droplet states has been observed.
- In 2019 at LENS (Florence) there was the observation of quantum droplets in a ⁸⁷Rb-⁴¹K heteronuclear bosonic mixture.⁵

⁵C. D'Errico *et al*, Phys. Rev. Res. **1**, 033155 (2019).



¹D.S. Petrov, Phys. Rev. Lett. **115**, 155302 (2015).

²C.R. Cabrera *et al.*, Science **359**, 6373 (2018).

³G. Semeghini et al.. Phys. Rev. Lett. **120**, 235301 (2018)

⁴P. Cheiney et al., Phys. Rev. Lett. 120 135301 (2018).

Beyond-mean-field effective action (I)

We consider a Bose gas made of atoms in two different hyperfine states. Each component can be described by a complex field $\psi_j(\mathbf{r},t)$ (j=1,2), whose dynamics results from the following real-time low-energy effective action

$$S = \int dt \, d^3 \mathbf{r} \Big[\sum_{j=1,2} \frac{i\hbar}{2} \left(\psi_j^* \partial_t \psi_j - \psi_j \partial_t \psi_j^* \right) - \mathcal{E}_{\text{tot}} (\psi_1, \psi_2) \Big] \,. \tag{1}$$

The total energy density \mathcal{E}_{tot} reads

$$\mathcal{E}_{\text{tot}} = \sum_{j=1,2} \left[\frac{\hbar^2}{2m} |\nabla \psi_j|^2 + V_{\text{ext}}(\mathbf{r}) |\psi_j|^2 + \frac{1}{2} g_{jj} |\psi_j|^4 \right] + g_{12} |\psi_1|^2 |\psi_2|^2 + \mathcal{E}_{\text{BMF}}(\psi_1, \psi_2) ,$$
(2)

where $V_{\rm ext}({\bf r})$ is an external confining potential, $g_{jk}=4\pi\hbar^2 a_{jk}/m$ are the interaction strengths with a_{jk} being intra- and inter-species scattering lengths, and $n_j({\bf r},t)=|\psi_j({\bf r},t)|^2$ is the number density of the species j.

Beyond-mean-field effective action (II)

The beyond-mean-field term $\mathcal{E}_{\mathsf{BMF}}$ arises from the zero-point energy of Bogoliubov elementary excitations⁶, namely

$$\mathcal{E}_{\text{BMF}} = \frac{8}{15\pi^2} \left(\frac{m}{\hbar^2}\right)^{3/2} (g_{11}n_1)^{5/2} f\left(\frac{g_{12}^2}{g_{11}g_{22}}, \frac{g_{22}n_2}{g_{11}n_1}\right)$$
(3)

with $f(x,y) = \sum_{\pm} [1 + y \pm \sqrt{(1-y)^2 + 4xy}]^{5/2} / (4\sqrt{2})$.

The calculation leading to the ground state properties can be simplified by assuming⁷ the two components occupying the same spatial mode $\phi(\mathbf{r},t)$. The bosonic fields can then be written as

$$\psi_j(\mathbf{r},t) = \sqrt{N_j} \ \phi(\mathbf{r},t) \ . \tag{4}$$

This assumption neglects the inter-component dynamics, resulting inadequate to probe spin-dipole oscillations.

⁷D.S. Petrov, Phys. Rev. Lett. **115**, 155302 (2015); D.S. Petrov and G.E. Astrakharcik, Phys. Rev. Lett. **117**, 100401 (2016).



⁶D.M. Larsen, Ann. Phys. 24, 89 (1963).

Beyond-mean-field effective action (III)

We work under the condition

$$\frac{N_1}{N_2} = \sqrt{\frac{a_{22}}{a_{11}}} \,, \tag{5}$$

which comes from the minimization of the mean-field energy density for the uniform system.⁸ By defining⁹

$$\Delta a = a_{12} + \sqrt{a_{11}a_{22}}$$
 with $a_{11} > 0, a_{22} > 0$, (6)

the total energy density reads (with $N=N_1+N_2$)

$$\frac{\mathcal{E}_{\text{tot}}}{N} = \frac{\hbar^{2}}{2m} |\nabla \phi|^{2} + V_{\text{ext}}(\mathbf{r}) |\phi|^{2} + N \frac{4\pi\hbar^{2}}{m} \frac{\Delta a \sqrt{a_{22}/a_{11}}}{\left(1 + \sqrt{a_{22}/a_{11}}\right)^{2}} |\phi|^{4} + N^{3/2} \frac{256\sqrt{\pi}\hbar^{2}}{15m} \left(\frac{\sqrt{a_{11}a_{22}}}{1 + \sqrt{a_{22}/a_{11}}}\right)^{5/2} f\left(\frac{a_{12}^{2}}{a_{11}a_{22}}, \sqrt{\frac{a_{22}}{a_{11}}}\right) |\phi|^{5}.$$
(7)

 $^{^{9}}$ Notice that the mean-field uniform system becomes unstable for $\Delta a < 0$.



⁸D.S. Petrov, Phys. Rev. Lett. **115**, 155302 (2015).

Beyond-mean-field effective action (IV)

Inspired by the experiment of the Tarruell group at Barcelona 10 with 39 K atoms, we study a quasi-1D optical waveguide which gives a harmonic confinement **only** on a transverse plane

$$V_{\text{ext}}(\mathbf{r}) = \frac{1}{2} m \omega_{\perp}^2 (x^2 + y^2)$$
 (8)

The presence of a harmonic potential defines a characteristic length scale, namely

$$a_{\perp} = \sqrt{\frac{\hbar}{m\omega_{\perp}}} \ . \tag{9}$$

We work with $a_{11} > 0$, $a_{22} > 0$ **but** $a_{12} < 0$. We investigate different values of

$$\Delta a = a_{12} + \sqrt{a_{11}a_{22}} \tag{10}$$

in the regime where $\Delta a < 0$, namely in the regime where $a_{12} < -\sqrt{a_{11}a_{22}}$.

¹⁰P. Cheiney et al., Phys. Rev. Lett. **120** 135301 (2018).



Gaussian ansatz and soliton-droplet crossover (I)

The properties of the system can be analytically explored by taking a Gaussian variational ansatz

$$\phi(\mathbf{r}) = \frac{1}{\pi^{3/4} \sigma_{\rho} \sigma_{z}^{1/2}} \exp\left(-\frac{x^{2} + y^{2}}{2\sigma_{\rho}^{2}} - \frac{z^{2}}{2\sigma_{z}^{2}}\right), \tag{11}$$

whose variational parameters are σ_{ρ} and σ_{z} .

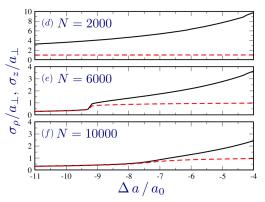
By replacing Eq. (11) in the total energy density, the variational energy per particle is then given by

$$\frac{E}{N\hbar\omega_{\perp}} = \frac{1}{4} \left(\frac{2}{\sigma_{\rho}^{2}} + \frac{1}{\sigma_{z}^{2}} \right) + \frac{\sigma_{\rho}^{2}}{2} + \frac{2N\Delta a}{\sqrt{2\pi}\sigma_{\rho}^{2}\sigma_{z}} \frac{\sqrt{a_{22}/a_{11}}}{\left(1 + \sqrt{a_{22}/a_{11}} \right)^{2}} + \frac{512\sqrt{2}}{75\sqrt{5}\pi^{7/4}} \frac{N^{3/2}}{(\sigma_{\rho}^{2}\sigma_{z})^{3/2}} \left(\frac{\sqrt{a_{11}a_{22}}}{1 + \sqrt{a_{22}/a_{11}}} \right)^{5/2} f\left(\frac{a_{12}^{2}}{a_{11}a_{22}}, \sqrt{\frac{a_{22}}{a_{11}}} \right). \tag{12}$$

Here the lengths are in units of a_{\perp} .

Experimentally, by means of Feshbach resonance, below a critical value of the external magnetic field, the condition $\Delta a < 0$ is achieved.

Gaussian ansatz and soliton-droplet crossover (II)



Axial width σ_z (black solid curve) and transverse width σ_ρ (red dashed curve) as a function of $\Delta a = a_{12} + \sqrt{a_{11}a_{22}}$, for three values of the particle number N in the $^{39}\text{K}-^{39}\text{K}$ mixture. We set $a_{11} = a_{22} = 33.5a_0$ and use $a_{12} < 0$. a_0 is the Bohr radius. The self-bound spherical droplet is obtained when $\sigma_\rho = \sigma_z.^{11}$

¹¹A. Cappellaro, T. Macri, and LS, Phys. Rev. A **97**, 053623 (2018).



Breathing modes (I)

A deeper insight into the differences between solitonic and droplet states can be reached by examining collective excitations around the minima of the Gaussian variational energy.

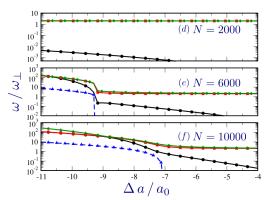
Thus, we adopt a time-dependent Gaussian variational ansatz for the complex scalar field:

$$\phi(\mathbf{r},t) = \sqrt{\frac{1}{\pi^{3/2} \prod_{K=x,y,z} \sigma_K(t)}} \exp\left(\sum_{K=x,y,z} \left(-\frac{K^2}{2\sigma_K(t)^2} + i\beta(t)K^2\right)\right).$$
(13)

Inserting this ansatz into the beyond-mean-field effective action, after some manipulations we obtain a new effective action for the three time-dependent variational widths $\sigma_K(t)$.

From the corresponding linearized Euler-Lagrange equations we find three collective frequencies: two (quasi-degenerate) frequencies are related to a breathing oscillation in the x-y plane while the third frequency is related to a beathing along the z axis.

Breathing modes (II)



Breathing frequencies (**red**, **green** and **black** curves) as a function of $\Delta a = a_{12} + \sqrt{a_{11}a_{22}}$, for three values of the particle number N in the $^{39}\text{K}-^{39}\text{K}$ mixture with $a_{11}=a_{22}=33.5a_0$ and $a_{12}<0$. The **blue curve** is $-\mu/\hbar$, with μ the chemical potential for $\mu<0$ (quantum droplet regime). 12

¹²A. Cappellaro, T. Macri, and LS, Phys. Rev. A **97**, 053623 (2018).



Spin-dipole mode (I)

We now consider the oscillatory motion occurring when there is a displacement $\bar{z}_1(t) - \bar{z}_2(t)$ along the z axis of the centers of mass $\bar{z}_j(t)$ of the two bosonic components.

To model this spin-dipole collective dynamics, we go beyond the assumption where the two components occupy the same spatial mode. The corresponding Gaussian variational ansatz is given by

$$\psi_{j}(\mathbf{r},t) = \sqrt{\frac{N_{j}}{\pi^{3/2} \prod_{K=x,y,z} \sigma_{K}(t)}} \times \exp \left[\sum_{K=x,y,z} \left(-\frac{(K - \bar{z}_{j}(t)\delta_{K,z})^{2}}{2\sigma_{K,j}(t)^{2}} + i\alpha_{j}(t)z + i\beta_{K,j}(t)K^{2} \right) \right],$$

where the fields $\psi_j(\mathbf{r},t)$ are normalized to N_j .

Spin-dipole mode (II)

We insert the time-dependent variational ansatz for the two fields $\psi_j(\mathbf{r},t)$ into the beyond-mean-field effective action, which becomes a functional of the variational parameters.

After several manipulations, we find that $\tilde{z}(t)=z_1(t)-z_2(t)$ satisfies the equation

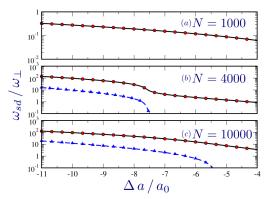
$$\frac{d^2}{dt^2}\tilde{z}(t) + \omega_{sd}^2\,\tilde{z}(t) = 0 , \qquad (14)$$

where the spin-dipole frequency ω_{sd} of the relative motion of the two bosonic clouds reads

$$\frac{\omega_{sd}^2}{\omega_{\perp}^2} = -\sqrt{\frac{8}{\pi}} \frac{N_1(a_{12}/a_{\perp})}{\sigma_{\rho,0}^2 \sigma_{z,0}^3} + \frac{2048}{25\pi^{1/4}} \frac{(a_{11}/a_{\perp})^{5/2} N_1^{3/2}}{\sigma_{\rho,0}^2 \sigma_{z,0}^{7/2}} , \qquad (15)$$

where $\sigma_{\rho,0}$ and $\sigma_{z,0}$ are the equilibrium values, and assuming $N_1 = N_2$.

Spin-dipole mode (III)



Spin-dipole frequency (black curve with red dots) as a function of $\Delta a = a_{12} + \sqrt{a_{11}a_{22}}$, for three values of the particle number N in the $^{39}\text{K}-^{39}\text{K}$ mixture with $a_{11}=a_{22}=33.5a_0$ and $a_{12}<0$. The **blue curve** is $-\mu/\hbar$ for $\mu<0.^{13}$

¹³A. Cappellaro, T. Macri, and LS, Phys. Rev. A **97**, 053623 (2018).



Bose-Bose droplets with Rabi coupling (I)

Let us consider again a bosonic gas is characterized by two-hyperfine components with bosonic complex fields $\psi_j({\bf r},t)$, j=1,2. The Lagrangian density of the system reads

$$\mathcal{L} = \sum_{j=1,2} \left[i\hbar \psi_j^* \partial_t \psi_j - \frac{\hbar^2}{2m} |\nabla \psi_j|^2 - \frac{g}{2} |\psi_j|^4 \right] - g_{12} |\psi_1|^2 |\psi_2|^2$$

$$+ \frac{\hbar \omega_R}{\hbar \omega_R} \left[\psi_1^* \psi_2 + \psi_2^* \psi_1 \right] . \tag{16}$$

In addition to the usual intra-species $(g=g_{11}=g_{22})$ and inter-species (g_{12}) contact interactions, atoms with mass m in different hyperfine states interact also via an external coherent Rabi coupling of frequency $\omega_{\mathbf{R}}$, which drives an exchange of atoms between the two components.

Bose-Bose droplets with Rabi coupling (II)

The presence of the Rabi coupling implies that only the total number

$$N = N_1(t) + N_2(t) \tag{17}$$

of atoms is conserved, with

$$N_j(t) = \int |\psi_j(\mathbf{r}, t)|^2 d^3\mathbf{r}$$
 (18)

the number of atoms in the j-th hyperfine component (j=1,2). In 2013 the existence and stability of the symmetric ground state with $N_1=N_2$ was discussed by Abad and Recati¹⁴. In addition to the symmetric configuration, a ground state with non-zero population imbalance is also possible¹⁵

¹⁵C. P. Search, A. G. Rojo, and P. R. Berman, Phys. Rev. A **64**, 013615 (2001).



¹⁴M. Abad and A. Recati, Eur. Phys. J. D **67** (7), 148 (2013).

Bose-Bose droplets with Rabi coupling (III)

At the mean-field level, for the **symmetric and uniform ground state**, characterized by $n_1 = n_2 = n/2$, the chemical potential μ reads

$$\mu = \frac{1}{2}gn(1+\epsilon) - \hbar\omega_{\rm R} , \qquad (19)$$

where $n = N/L^2$ is the 2D total number density of bosons, with g > 0 and $\epsilon = g_{12}/g$.

This mean-field **symmetric and uniform ground state** is stable under the conditions¹⁶

$$g + g_{12} > 0$$
 and $(g - g_{12})n + 2\hbar\omega_{\rm R} > 0$, (20)

namely

$$-1 < \epsilon < 1 + \frac{2\hbar\omega_{\rm R}}{gn} \ . \tag{21}$$

¹⁶M. Abad and A. Recati, Eur. Phys. J. D **67** (7), 148 (2013).



Bose-Bose droplets with Rabi coupling (IV)

The Bogoliubov spectrum of elementary excitations of the uniform system has two branches, given by

$$E_k^{(-)} = \sqrt{\frac{\hbar^2 k^2}{2m} \left[\frac{\hbar^2 k^2}{2m} + 2(\mu + \hbar \omega_{\rm R}) \right]},$$
 (22)

$$E_k^{(+)} = \sqrt{\frac{\hbar^2 k^2}{2m}} \left[\frac{\hbar^2 k^2}{2m} + 2A \right] + B , \qquad (23)$$

where μ is the chemical potential. Moreover the two parameters appearing in the gapped branch are

$$A = \frac{1}{2}gn(1-\epsilon) + 2\hbar\omega_{\rm R} , \qquad (24)$$

$$B = 4\hbar\omega_{\rm R} \left[\frac{1}{2} g n (1 - \epsilon) + \hbar\omega_{\rm R} \right] , \qquad (25)$$

again with $\epsilon = g_{12}/g$ and $g = g_{11} = g_{22}$.

Bose-Bose droplets with Rabi coupling (V)

Remember that we set

$$\epsilon = g_{12}/g = a_{12}/a$$
 (26)

with a the intra-component scattering length and a_{12} the inter-component scattering length. We also introduce

$$\bar{n} = na^3 \tag{27}$$

that is our gas parameter. We use the elementary excitations to obtain the beyond-mean-field one-loop (Gaussian) corrections to the mean-field energy. In particular, we find 17

$$\frac{\mathcal{E}}{E_B/a^3} = \pi (1+\epsilon) \bar{n}^2 - \bar{\omega}_R \bar{n} + \frac{8}{15\pi^2} [2\pi \bar{n} (1+\epsilon)]^{5/2}
+ \frac{8}{15\pi^2} [2\pi \bar{n} (1-\epsilon)]^{5/2} + \frac{14}{3\pi^2} \bar{\omega}_R [2\pi \bar{n} (1-\epsilon)]^{3/2}, (28)$$

where $E_B = \hbar^2/ma^2$ and $\bar{\omega}_R = \hbar\omega_R/E_B$.

¹⁷A. Cappellaro, T. Macri, G. F. Bertacco, and LS, Sci. Rep. **7**, 13358 (2017).



Bose-Bose droplets with Rabi coupling (VI)

Notice that in Eq. (28) for $\bar{\omega}_R=0$ one recovers the Larsen's zero-temperature equation of state¹⁸

Moreover, from Eq. (28) one finds that for $|\epsilon|>1$ the uniform configuration is not stable. If $\epsilon>1$, at the mean field level, one expects phase separation or population imbalance.¹⁹

Instead, if $\epsilon<-1$ the term proportional to $[(1+\epsilon)\bar{n}]^{5/2}$ becomes imaginary and it gives rise to a dissipative dynamics. However, this dissipative term can be neglected if \bar{n} is not too large. In this way, by minimization of the energy density of Eq. (28), for $\epsilon<-1$ we obtain

$$\bar{n}_{\min} = \left(\frac{5\sqrt{\pi}|1+\epsilon|}{32\sqrt{2}(1+|\epsilon|)^{5/2}} \left[1+\sqrt{1-\frac{1792\bar{\omega}_R}{15\pi^2}\frac{(1+|\epsilon|)^4}{|1+\epsilon|^2}}\right]\right)^2 . \quad (29)$$

as a local minimum, if $\bar{\omega}_R < \bar{\omega}_c = \frac{15\pi^2}{1792} \frac{|1+\epsilon|^2}{(1+|\epsilon|)^4}$. For larger $\bar{\omega}_R$ there is only the absolute minimum with zero energy at $\bar{n}=0$.

¹⁹K. Furutani, A. Perali, and LS, Phys. Rev. A **107**, L041302 (2023).



¹⁸D.M. Larsen, Ann. Phys. 24, 89 (1963).

Bose-Bose droplets with Rabi coupling (VII)

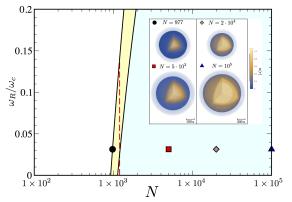
The local minimum \bar{n}_{\min} we found for $\epsilon < -1$ and $\bar{\omega}_R < \bar{\omega}_c = \frac{15\pi^2}{1792} \frac{|1+\epsilon|^2}{(1+|\epsilon|)^4}$ suggests the existence of a Bose-Bose-droplet also in the presence of Rabi coupling.

To get more info about this Bose-Bose droplet with Rabi we consider a space-time dependent complex field $\Psi(\mathbf{r},t)$ such that $n(\mathbf{r},t) = \left|\Psi(\mathbf{r},t)\right|^2$ is the space-time dependent total number density, and clearly $N = \int d^3\mathbf{r} \ n(\mathbf{r},t)$. The dynamics of $\Psi(\mathbf{r},t)$ is driven by the following real-time effective action

$$S_{\text{eff}}[\Psi^*, \Psi] = \int dt \, d^3 \mathbf{r} \left[i\hbar \Psi^* \partial_t \Psi - \frac{\hbar^2 |\nabla \Psi|^2}{2m} - \mathcal{E}_{ND}(|\Psi|^2) \right], \quad (30)$$

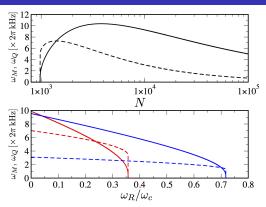
where the non-dissipative (ND) energy density \mathcal{E}_{ND} is derived from Eq. (28) neglecting the imaginary term proportional to $[(1+\epsilon)\bar{n}]^{5/2}$.

Bose-Bose droplets with Rabi coupling (VIII)



Phase diagram for the Bose-Bose droplet with Rabi coupling: unstable (white region), metastable (yellow region), stable (light blue region). We choose ^{39}K atoms with $N_1=N_2=N$, $a=40a_0$ with a_0 the Bohr radius, and $\epsilon=-1.5$, which implies $\omega_c/(2\pi)=31.8$ kHz. Red dashed line refers to a system of N=1200 particles.

Bose-Bose droplets with Rabi coupling (IX)



Upper panel: monopole (breathing) mode frequency ω_M (solid) and quadrupole mode frequency ω_Q (dashed) of the Bose-Bose droplet as a function of particle number, with $\omega_R/2\pi=1\,\mathrm{kHz},~a=40a_0$, and $\epsilon=-1.5$. Below $N\simeq 977$ the droplet becomes unstable. Lower panel: frequencies as a function of Rabi coupling for $N=2\cdot 10^3$ (red), and $N=10^5$ (blue). The critical Rabi frequency occurs at $\omega_c/(2\pi)=31.8\,\mathrm{kHz}$.

Conclusions

- We have analyzed a two-component BEC with attractive interparticle interactions along the crossover from soliton to self-bound droplets in a quasi one dimensional waveguide.
- We have found a sharp difference of the collective modes in the two regimes:
 - in the soliton regime: two distinguishable collective frequencies;
 - deep into the spherical droplet regime: only one breathing frequency (triple degenerate).
- Also the spin-dipole mode can be excited, and we have found an analytical formula for it.
- The inclusion of a Rabi coupling modifies the properties of the two-component quantum droplet.

Acknowledgements

Thank you for attention!

Main sponsors:

- MUR-PRIN 2022 "Quantum Atomic Mixtures: Droplets, Topological Structures, and Vortices".
- INFN Iniziativa Specifica "Quantum".
- Dipartimenti di Eccellenza MUR 2023-2027 "Frontiere Quantistiche",
 DFA UNIPD.
- National Center for HPC, the Big Data and Quantum Computing [Spoke 10: Quantum Computing].