First and second sound in two-dimensional bosonic and fermionic systems

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Summary

- Landau theory of first and second sound
- Weakly-interacting 2D Bose gas
- 2D Fermi gas in the BCS-BEC crossover
- Conclusions

Landau theory of first and second sound (I)

According to Landau's two fluid theory[1] of **superfluids** the total number density n of a system in the superfluid phase can be written as

$$n = n_s + n_n \ , \qquad (1)$$

where n_s is the superfluid density and n_n is the normal density. At the critical temperature T_c one has $n_n = n$ and, correspondingly, $n_s = 0$.

Following Landau, in a **superfluid** a local perturbation excites two wave-like modes - first and second sound - which propagate with velocities u_1 and u_2. These velocities are determined by the positive solutions of the algebraic biquadratic equation

$$u^4 + (c_{10}^2 + c_{20}^2)u^2 + c_T^2 c_{20}^2 = 0 \ . \qquad (2)$$

The first sound u_1 is the largest of the two positive roots of Eq. (2) while the second sound u_2 is the smallest positive one.

[1] L.D. Landau, J. Phys. (USSR) **5**, 71 (1941).

Landau theory of first and second sound (II)

In the biquadratic equation there is the adiabatic sound velocity

$$c_{10} = v_A = \sqrt{\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_{\bar{S},V}} \tag{3}$$

with $\bar{S} = S/N$ the entropy per particle, the entropic sound (or Landau) velocity,

$$c_{20} = v_L = \sqrt{\frac{1}{m}\frac{\bar{S}^2}{\left(\frac{\partial \bar{S}}{\partial T}\right)_{N,V}}\frac{n_s}{n_n}} \tag{4}$$

with n_s/n_n the ratio between superfluid and normal density, and the isothermal sound velocity

$$c_T = v_T = \sqrt{\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_{T,V}}\,. \tag{5}$$

Weakly-interacting 2D Bose gas (I)

The Helmholtz free energy[2] of a weakly-interacting two-dimensional gas of identical bosons of mass m can be written as $(\hbar = k_{\mathrm{B}} = 1)$

$$F = F_0 + F_{\mathrm{Q}} + F_T = \frac{g}{2}\frac{N^2}{L^2} + \frac{1}{2}\sum_{\mathbf{p}} E_p + T\sum_{\mathbf{p}} \ln\left[1 - e^{-E_p/T}\right], \quad (6)$$

where F_0 is the mean-field zero-temperature free energy with g is the Bose-Bose interaction strength, N is the total number of identical bosons confined in a square of area L^2 and $n = N/L^2$ is the two-dimensional number density. F_T is the low-temperature free energy with T is the absolute temperature and

$$E_p = \sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m} + 2gn\right)}, \quad (7)$$

is the Bogoliubov spectrum.

[2] K. Furutani, A. Tononi, and LS, New J. Phys. **23** 043043 (2021).

Weakly-interacting 2D Bose gas (II)

The quantum correction F_{Q} in the free energy is obviously ultraviolet divergent and requires a regularization procedure. Dimensional regularization[3] leads to

$$F_{\mathrm{Q}} = -L^2 \frac{m}{8\pi} \left[\ln\left(\frac{\epsilon_\Lambda}{gn}\right) - \frac{2}{\eta} \right] (gn)^2 \ , \tag{8}$$

where $\epsilon_\Lambda = 4e^{-2\gamma-1/2}/\left(ma_{2\mathrm{D}}^2\right)$ is a cutoff energy, $\gamma = 0.577$ is the Euler-Mascheroni constant, $a_{2\mathrm{D}}$ is the 2D s-wave scattering length, and

$$\eta = \frac{mg}{2\pi} \tag{9}$$

is the adimensional gas parameter.[4] Moreover, one also finds

$$\frac{\epsilon_\Lambda}{gn} = \frac{2\pi}{N} \frac{e^{-2\gamma-1/2+2/\eta}}{\eta} \ . \tag{10}$$

[3] LS and F. Toigo, Phys. Rep. **640**, 1 (2016)

[4] K. Furutani, A. Tononi, and LS, New J. Phys. **23** 043043 (2021).

Weakly-interacting 2D Bose gas (III)

All the thermodynamic quantities can be obtained from the Helmholtz free energy of Eq. (6). For instance, the pressure P is given by

$$P = -\left(\frac{\partial F}{\partial L^2}\right)_{N,T} \tag{11}$$

while the the entropy reads

$$S = \left(\frac{\partial}{\partial T}\frac{F}{N}\right)_{N,L^2} . \tag{12}$$

Instead, the normal density n_n can be extracted from the Landau formula

$$n_n = -\int \frac{d^2\mathbf{p}}{(2\pi)^2}\,\frac{p^2}{2m}\,\frac{df_{\rm B}(E_p)}{dE_p} , \tag{13}$$

where $f_{\rm B}(E) = 1/\left(e^{E/T} - 1\right)$ is the Bose-Einstein distribution.

Weakly-interacting 2D Bose gas (IV)

Actually, the Landau formula for the normal density does not take into account the formation of quantized vortices and anti-vortices by increasing the temperature. These quantized vortices are crucial for the 2D Bose gas to obtain the phenomenology predicted by Berezinskii[5] and Kosterlitz-Thouless.[6]

The presence of quantized vortices renormalize the superfluid density $n_s = n - n_n$. The renormalized superfluid density $n_s(t = +\infty)$ is obtained by solving the Nelson-Kosterlitz-Nelson renormalization group equations[7]

$$\begin{aligned} \partial_t K^{-1}(t) &= 4\pi^3 y^2(t) \\ \partial_t y(t) &= [2 - \pi K(t)]\, y(t) \end{aligned} \tag{14}$$

where $K(t) = n_{\rm s}(t)/T$, with $n_{\rm s}(t)$ the superfluid density at the adimensional fictitious time t, and $y(t) = \exp\left[-\mu_{\rm c}(t)/T\right]$ is the fugacity, where $\mu_{\rm c}(t)$ is the vortex chemical potential at fictitious time t.

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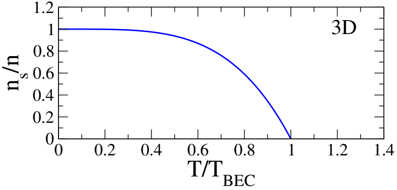
[5] V.L. Berezinskii, *Sov. Phys. JETP* **34**, 610 (1972).

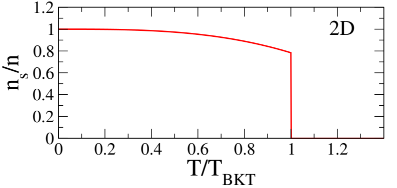
[6] J.M. Kosterlitz and D.J. Thouless D J. Phys. C **5**, L124 (1972).

[7] D.R. Nelson and J.M. Kosterlitz, Phys. Rev. Lett. **39**, 1201 (1977).

Weakly-interacting 2D Bose gas (V)

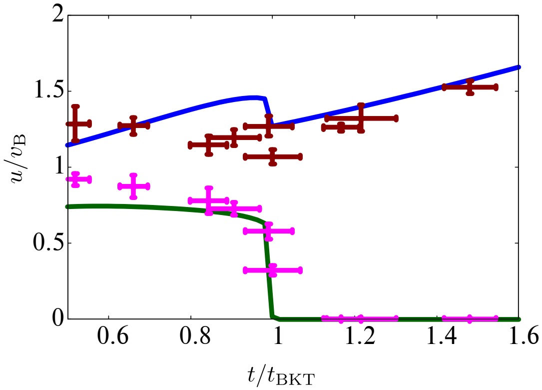
For 3D superfluids the transition to the normal state is a BEC phase transition, while in 2D superfluids the transition to the normal state is something different: a topological phase transition.





An important prediction of the Kosterlitz-Thouless transition is that, contrary to the 3D case, in 2D the superfluid fraction n_s/n jumps to zero above the Berezinskii-Kosterlitz-Thhouless critical temperature T_{BKT}.

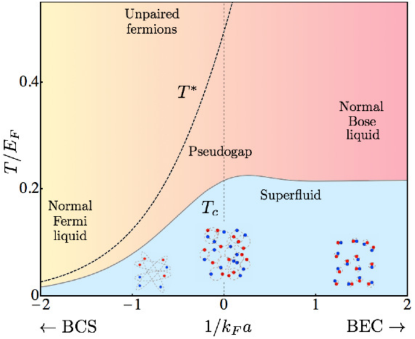
Weakly-interacting 2D Bose gas (VI)



First sound and second sound velocity vs adimensional temperature plotted in comparison with **recent experimental data** near T_{BKT} [P. Christodoulou *et al.*, Nature **594**, 191 (2021)]. Here $v_B = gn$ is the Bogoliubov velocity, $N = 2178$ and $\eta = 0.102$. The **blue line** is our **first sound velocity** u_1 while the **green line** is our **second sound velocity** u_2. Figure adapted from K. Furutani, A. Tononi, and LS, New J. Phys. **23**, 043043 (2021).

2D Fermi gas in the BCS-BEC crossover (I)

In 2004 the 3D BCS-BEC crossover has been observed with **ultracold gases made of two-component fermionic ^{40}K or ^{6}Li atoms.**[8]



This crossover is obtained using a Fano-Feshbach resonance to change the 3D s-wave scattering length a_F of the inter-atomic potential.

[8] C.A. Regal et al., PRL **92**, 040403 (2004); M.W. Zwierlein et al., PRL **92**, 120403 (2004); J. Kinast et al., PRL **92**, 150402 (2004).

2D Fermi gas in the BCS-BEC crossover (II)

More recently also the 2D BEC-BEC crossover has been achieved experimentally[9] with a **Fermi gas of two-component ^{6}Li atoms**.

Contrary to the 3D case, 2D realistic interatomic attractive potentials have always a bound state. In particular[10], the binding energy $\epsilon_B > 0$ of two fermions is related to the 2D scattering length a_F by

$$\epsilon_B = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{m{a_F}^2} \, , \tag{15}$$

where $\gamma = 0.577$ is the Euler-Mascheroni constant. Moreover, the attractive (negative) interaction strength $\mathbf{g}$ of s-wave pairing is related to the binding energy by the expression[11]

$$-\frac{1}{\mathbf{g}} = \frac{1}{2L^2} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 k^2}{2m} + \frac{1}{2}\epsilon_B} \, . \tag{16}$$

[9] V. Makhalov et al. PRL **112**, 045301 (2014); M.G. Ries et al., PRL **114**, 230401 (2015); I. Boettcher et al., PRL **116**, 045303 (2016); K. Fenech et al., PRL **116**, 045302 (2016).

[10] C. Mora and Y. Castin, 2003, PRA **67**, 053615.

[11] M. Randeria, J-M. Duan, and L-Y. Shieh, PRL **62**, 981 (1989).

2D Fermi gas in the BCS-BEC crossover (III)

To study the 2D BCS-BEC crossover we adopt the formalism of functional integration[12]. The partition function $\mathcal{Z}$ of the uniform system with fermionic fields $\psi_s(\mathbf{r},\tau)$ at temperature T, in a 2-dimensional volume L^2, and with chemical potential μ reads

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \ \exp\left\{ -\frac{S}{\hbar} \right\} , \tag{17}$$

where ($\beta \equiv 1/(k_B T)$ with k_B Boltzmann's constant)

$$S = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r} \ \mathcal{L} \tag{18}$$

is the Euclidean action functional with Lagrangian density

$$\mathcal{L} = \bar{\psi}_s \left[\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \psi_s + \mathbf{g}\, \bar{\psi}_\uparrow \, \bar{\psi}_\downarrow \, \psi_\downarrow \, \psi_\uparrow \tag{19}$$

where $\mathbf{g}$ is the attractive strength ($\mathbf{g} < 0$) of the s-wave coupling.

[12] N. Nagaosa, Quantum Field Theory in Condensed Matter (Springer, 1999).

2D Fermi gas in the BCS-BEC crossover (IV)

In particular, we are interested in the grand potential Ω, given by

$$\Omega = -\frac{1}{\beta} \ln(\mathcal{Z}) \simeq -\frac{1}{\beta} \ln(\mathcal{Z}_{mf} \mathcal{Z}_g) = \Omega_{mf} + \Omega_g \ , \tag{20}$$

where

$$\mathcal{Z}_{mf} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp\left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar} \right\} \tag{21}$$

is the mean-field partition function and

$$\mathcal{Z}_g = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \mathcal{D}[\eta, \bar{\eta}] \ \exp\left\{ -\frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar} \right\} \tag{22}$$

is the partition function of Gaussian pairing fluctuations.

2D Fermi gas in the BCS-BEC crossover (V)

After functional integration over quadratic fields, one finds that the mean-field grand potential reads[13]

$$\Omega_{mf} = -\frac{\Delta_0^2}{\mathbf{g}} L^2 + \sum_{\mathbf{k}} \left(\frac{\hbar^2 k^2}{2m} - \mu - E_{sp}(\mathbf{k}) - \frac{2}{\beta} \ln \left(1 + e^{-\beta\, E_{sp}(\mathbf{k})}\right) \right) \quad (23)$$

where

$$E_{sp}(\mathbf{k}) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu \right)^2 + \Delta_0^2} \quad (24)$$

is the spectrum of fermionic single-particle excitations.

[13] A. Altland and B. Simons, Condensed Matter Field Theory (Cambridge Univ. Press, 2006).

2D Fermi gas in the BCS-BEC crossover (VI)

The Gaussian grand potential is instead given by

$$\Omega_g = \frac{1}{2\beta} \sum_Q \ln \det(\mathbf{M}(Q)) \ , \tag{25}$$

where $\mathbf{M}(Q)$ is the inverse propagator of Gaussian fluctuations of pairs and $Q = (\mathbf{q}, i\Omega_m)$ is the 4D wavevector with $\Omega_m = 2\pi m/\beta$ the Matsubara frequencies and $\mathbf{q}$ the 3D wavevector.[14]
The sum over Matsubara frequencies is quite complicated and it does not give a simple expression. An approximate formula[15] is

$$\Omega_g \simeq \frac{1}{2} \sum_{\mathbf{q}} E_{col}(\mathbf{q}) + \frac{1}{\beta} \sum_{\mathbf{q}} \ln\left(1 - e^{-\beta\, E_{col}(\mathbf{q})}\right) \ , \tag{26}$$

where

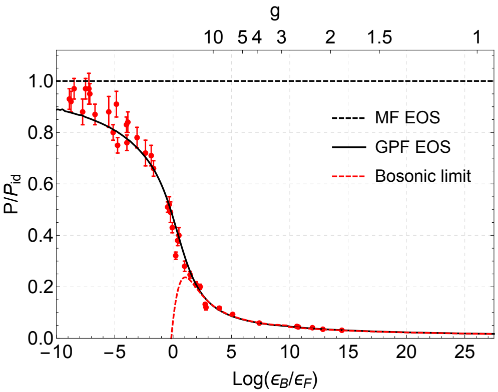
$$E_{col}(\mathbf{q}) = \hbar\, \omega(\mathbf{q}) \tag{27}$$

is the spectrum of bosonic collective excitations with $\omega(\mathbf{q})$ derived from

$$\det(\mathbf{M}(\mathbf{q}, \omega)) = 0 \ . \tag{28}$$

[14] R.B. Diener, R. Sensarma, M. Randeria, Phys. Rev. A **77**, 023626 (2008).
[15] E. Taylor, A. Griffin, N. Fukushima, Y. Ohashi, Phys. Rev. A **74**, 063626 (2006).

2D Fermi gas in the BCS-BEC crossover (VII)



Zero-temperature scaled pressure P/P_{id} vs scaled binding energy ϵ_B/ϵ_F. Notice that $P = -\Omega/L^2$ and P_{id} is the pressure of the ideal 2D Fermi gas. Filled circles with error bars: experimental data of Makhalov *et al.* [16]. Solid line: our regularized Gaussian theory.[17] Figure adapted from G. Bighin and LS, Phys. Rev. B **93**, 014519 (2016).

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[16] V. Makhalov et al. Phys. Rev. Lett. **112**, 045301 (2014).
[17] See also L. He, H. Lu, G. Cao, H. Hu and X.-J. Liu, PRA **92**, 023620 (2015).

2D Fermi gas in the BCS-BEC crossover (VIII)

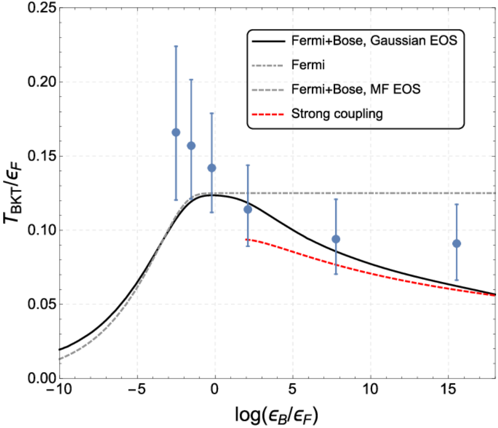
We are now interested on the temperature dependence of superfluidy density $n_s(T)$ of the system.
At the Gaussian level $n_s(T)$ depends only on fermionic single-particle excitations $E_{sp}(k)$.[18] **Beyond the Gaussian level** also bosonic collective excitations $E_{col}(q)$ contribute.[19]
Thus, we assume the following Landau-type formula for the superfluid density[20]

$$n_s(T) = n - \beta \int \frac{\mathrm{d}^2 k}{(2\pi)^2} k^2 \frac{e^{\beta E_{sp}(k)}}{(e^{\beta E_{sp}(k)} + 1)^2} - \frac{\beta}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} q^2 \frac{e^{\beta E_{col}(q)}}{(e^{\beta E_{col}(q)} - 1)^2} . \tag{29}$$

Clearly, this bare superfluid density must be renormalized using the flow equations of Kosterlitz-Thouless-Nelson, which take into account the effect of **quantized vortices and anti-vortices**.

[18] E. Babaev and H.K. Kleinert, Phys. Rev. B **59**, 12083 (1999).
[19] L. Benfatto, A. Toschi, and S. Caprara, Phys. Rev. B **69**, 184510 (2004).
[20] G. Bighin and LS, Phys. Rev. B **93**, 014519 (2016).

2D Fermi gas in the BCS-BEC crossover (IX)



Our theoretical predictions[21] for the Berezinskii-Kosterlitz-Thouless critical temperature T_{BKT} compared to experimental observation[22] (filled circles with error bars).

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[21] G. Bighin and LS, Phys. Rev. B **93**, 014519 (2016).
[22] P.A. Murthy et al., Phys. Rev. Lett. **115**, 010401 (2015).

2D Fermi gas in the BCS-BEC crossover (X)

We calculate the first sound velocity u_1 and the second sound velocity u_2 in the 2D BCS-BEC crossover.
We also analyze the amplitudes modes W_1 and W_2 of the response to a density perturbation,[23] i.e.

$$\delta n(x,t) = W_1 \delta n_1(x \pm u_1 t) + W_2 \delta n_2(x \pm u_2 t) \tag{30}$$

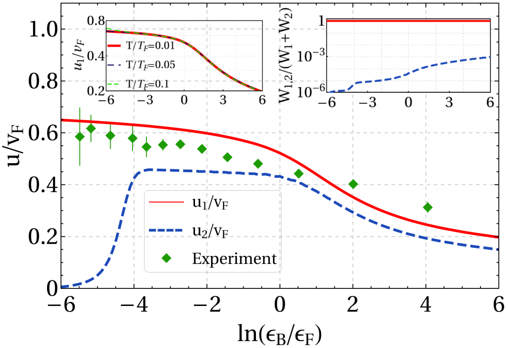
where

$$\frac{W_1}{W_1 + W_2} = \frac{\left(u_1^2 - c_{20}^2\right) u_2^2}{\left(u_1^2 - u_2^2\right) c_{20}^2} \tag{31}$$

and

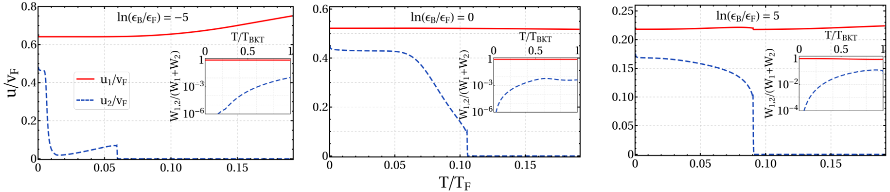
$$\frac{W_2}{W_1 + W_2} = \frac{\left(c_{20}^2 - u_2^2\right) u_1^2}{\left(u_1^2 - u_2^2\right) c_{20}^2} \; . \tag{32}$$

[23] T. Ozawa and S. Stringari, Phys. Rev. Lett. **112**, 025302 (2014).



First sound velocity u_1 (**red solid line**) and **second sound** velocity u_2 (**blue dashed line**) along the BCS-BEC crossover, at temperature $T/T_F = 0.01$, with $T_F = \epsilon_F/k_B$ and $v_F = \sqrt{2\epsilon_F/m}$. **Green points**: recent measurements of the first sound [M. Bohlen *et al.* Phys. Rev. Lett. **124**, 240403 (2020).] Right inset: relative contribution to the density response of u_1 (red solid line) and u_2 (blue dashed line). Figure adapted from A. Tononi, A. Capellaro, G. Bighin, and LS, Phys. Rev. A **103**, L061303 (2021).

2D Fermi gas in the BCS-BEC crossover (XII)



First sound velocity u_1/v_F (**red solid line**) and **second sound** velocity u_2/v_F (**blue dashed line**) plotted in terms of the rescaled temperature T/T_F, for three different values of the crossover parameter: $\ln(\epsilon_B/\epsilon_F) = -5$ (BCS regime), $\ln(\epsilon_B/\epsilon_F) = 0$ (unitary regime), and $\ln(\epsilon_B/\epsilon_F) = 5$ (BEC regime). Insets: relative contribution to the density responses $W_{1,2}/(W_1 + W_2)$ of u_1 and u_2. Figure adapted from A. Tononi, A. Capellaro, G. Bighin, and LS, Phys. Rev. A **103**, L061303 (2021).

Conclusions

- First and second sound of bosonic and fermionic superfluids can be derived knowing the equation of state and the superfluid fraction and using the Landau's two-fluid theory.
- In the case of a **2D weakly-interacting Bose gas**, we have calculated first and second sound. The comparison with recent measurement near T_{BKT} is quite good.
- In the **BCS-BEC crossover of the 2D Fermi gas**, to get a good agreement with experimental data for the equation of state, the critical temperature T_{BKT}, and the sound modes, both fermionic single-particle excitations and bosonic collective excitations are needed.

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