Superfluid fraction and Sound Velocity in the 2D BCS-BEC Crossover

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Como, June 5, 2014

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Phys. Rev. A. **88**, 053612 (2013).

Summary

- Formalism for 2D Fermi superfluids
- Mean-field results in the BCS-BEC crossover
- Phase fluctuations and superfluid fraction
- Phase and amplitude fluctuations and sound velocity
- Open problems

Formalism for 2D Fermi superfluids (I)

We consider a two-dimensional Fermi gas of ultracold and dilute two-spin-component neutral atoms. We adopt the path integral formalism, where the atomic fermions are described by the complex Grassmann fields $\psi_s(\mathbf{r},\tau)$, $\bar{\psi}_s(\mathbf{r},\tau)$ with spin $s = (\uparrow, \downarrow)$. The partition function $\mathcal{Z}$ of the uniform system at temperature T, in a two-dimensional volume L^2, and with chemical potential μ can be written as

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \ \exp\left\{ -\frac{1}{\hbar} \ S \right\} , \tag{1}$$

where

$$S = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r} \ \mathscr{L} \tag{2}$$

is the Euclidean action functional and

Formalism for 2D Fermi superfluids (II)

$\mathscr{L}$ is the Euclidean Lagrangian density, given by

$$\mathscr{L} = \bar{\psi}_s \left[\hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + g \, \bar{\psi}_\uparrow \, \bar{\psi}_\downarrow \, \psi_\downarrow \, \psi_\uparrow \tag{3}$$

where g is the strength of the s-wave inter-atomic coupling ($g < 0$ in the BCS regime). Summation over the repeated index s in the Lagrangian is meant and $\beta \equiv 1/(k_B T)$ with k_B Boltzmann's constant.

Formalism for 2D Fermi superfluids (III)

Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathscr{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(\mathbf{r},\tau)$ so that:

$$\mathcal{Z} = \int \mathcal{D}[\psi_s,\bar{\psi}_s]\,\mathcal{D}[\Delta,\bar{\Delta}]\ \exp\left\{-\frac{S_e(\psi_s,\bar{\psi}_s,\Delta,\bar{\Delta})}{\hbar}\right\}, \qquad (4)$$

where

$$S_e(\psi_s,\bar{\psi}_s,\Delta,\bar{\Delta}) = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r}\ \mathscr{L}_e(\psi_s,\bar{\psi}_s,\Delta,\bar{\Delta}) \qquad (5)$$

and the (exact) effective Euclidean Lagrangian density $\mathscr{L}_e(\psi_s,\bar{\psi}_s,\Delta,\bar{\Delta})$ reads

$$\mathscr{L}_e = \bar{\psi}_s\left[\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu\right]\psi_s + \bar{\Delta}\,\psi_\downarrow\,\psi_\uparrow + \Delta\bar{\psi}_\uparrow\,\bar{\psi}_\downarrow - \frac{|\Delta|^2}{g}\,. \qquad (6)$$

Formalism for 2D Fermi superfluids (IV)

We want to investigate the effect of fluctuations of the gap field $\Delta(\mathbf{r}, t)$ around its mean-field value Δ_0 which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r}, \tau) = (\Delta_0 + \sigma(\mathbf{r}, \tau))\ e^{i\theta(\mathbf{r}, \tau)}\ , \tag{7}$$

where $\theta(\mathbf{r}, \tau)$ is the phase of the gap field (it describes the Goldstone field of the U(1) symmetry) and $\sigma(\mathbf{r}, \tau)$ describes amplitude fluctuations. The adopted polar representation for $\Delta(\mathbf{r}, t)$ automatically satisfies Goldstone's theorem.

Mean-field results in the BCS-BEC crossover (I)

By neglecting both phase and amplitude fluctuations, i.e. by setting $\theta(\mathbf{r}, t) = 0$ and $\sigma(\mathbf{r}, \tau) = 0$, and integrating over the fermionic fields one gets immediately the mean-field partition function

$$\mathcal{Z}_{mf} = \exp\left\{-\frac{S_{mf}}{\hbar}\right\} = \exp\left\{-\beta\,\Omega_{mf}\right\} , \tag{8}$$

where

$$\frac{S_{mf}}{\hbar} = -Tr[\ln\left(G_0^{-1}\right)] - \beta L^2 \frac{\Delta_0^2}{g} \tag{9}$$

and explicitly

$$\frac{S_{mf}}{\hbar} = -\sum_{\mathbf{k}} \left[2\ln\left(2\cosh\left(\beta E_k/2\right)\right) - \beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)\right] - \beta L^2 \frac{\Delta_0^2}{g} , \tag{10}$$

with

$$G_0^{-1} = \begin{pmatrix} \hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu & \Delta_0 \\ \Delta_0 & \hbar\partial_\tau + \frac{\hbar^2}{2m}\nabla^2 + \mu \end{pmatrix} \quad (11)$$

the inverse mean-field Green function, and

$$E_k = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2} \quad (12)$$

the energy of the fermionic elementary excitations.

Mean-field results in the BCS-BEC crossover (III)

The constant, uniform and real gap parameter Δ_0 can be obtained by minimizing Ω_{mf} :

$$\frac{\partial \Omega_{mf}(\Delta_0)}{\partial \Delta_0} = 0 \tag{13}$$

which gives the familiar gap equation

$$-\frac{1}{g} = \frac{1}{L^2} \sum_{\mathbf{k}} \frac{\tanh\left(\beta E_k/2\right)}{2E_k} \,. \tag{14}$$

The integral on the right hand side of this equation is formally divergent. Nevertheless this divergence is easily removed. Contrary to the 3D case, in 2D a bound-state energy ϵ_B exists for any value of the attractive interaction strength g between atoms. By expressing the bare interaction strength g in terms of the physical binding energy ϵ_B through

$$-\frac{1}{g} = \frac{1}{L^2} \sum_{\mathbf{k}} \frac{1}{2\frac{\hbar^2 k^2}{2m} + \epsilon_B} \,. \tag{15}$$

we obtain the regularized gap equation

$$\sum_{\mathbf{k}} \left(\frac{\tanh\left(\beta E_k/2\right)}{2E_k} - \frac{1}{2\frac{\hbar^2 k^2}{2m} + \epsilon_B} \right) = 0 \,. \tag{16}$$

Mean-field results in the BCS-BEC crossover (IV)

The total number N of fermions is obtained from the familiar thermodynamic relation

$$N = - \left(\frac{\partial \Omega_{mf}}{\partial \mu} \right)_{L^2, T} , \quad (17)$$

which gives the number equation

$$N = \sum_{\mathbf{k}} \left(1 \quad - \quad \frac{\hbar^2 k^2 / 2m - \mu}{E_k} \tanh \left(\beta E_k / 2 \right) \right) \quad (18)$$

which must be solved together with (16) to determine the behavior of Δ_0 and μ as a function of the temperature T and of the binding energy ϵ_B at fixed number density $n = N/L^2$.

Mean-field results in the BCS-BEC crossover (V)

At zero temperature ($T = 0$) one easily finds the exact solutions of Eqs. (16) and (18) as

$$\mu = \epsilon_F - \frac{1}{2}\epsilon_B \quad \text{at } T = 0 \,, \tag{19}$$

$$\Delta_0 = \sqrt{2\epsilon_F \epsilon_B} \quad \text{at } T = 0 \,. \tag{20}$$

We identify the temperature T^* as the temperature at which the mean-field energy gap Δ_0 becomes. After some manipulations one obtains the equations determining T^* as a function of n (through the 2D Fermi energy $\epsilon_F = (\hbar/m)\pi n$) and the binding energy ϵ_B:

$$\mu(T^*) = k_B T^* \, \ln\left(e^{\epsilon_F/(k_B T^*)} - 1\right) \,, \tag{21}$$

$$\epsilon_B = k_B T^* \, \frac{\pi}{\gamma} \exp\left(- \int_0^{\mu(T^*)/(2k_B T^*)} \frac{\tanh(u)}{u} \, du \right) \,, \tag{22}$$

where $\gamma = 1.781$.

Mean-field results in the BCS-BEC crossover (VI)

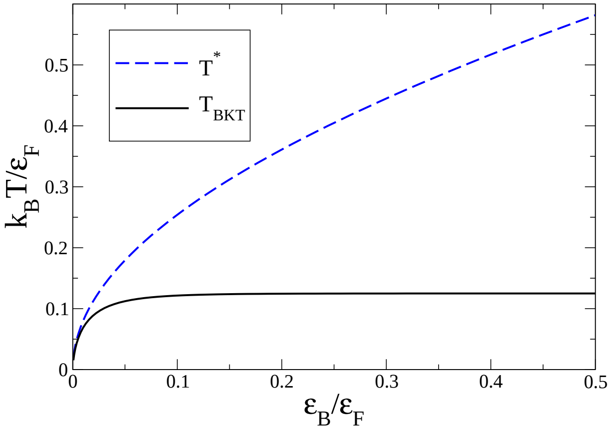

Figure: Relevant temperatures of the Fermi gas as a function of the scaled binding energy ϵ_B/ϵ_F, with ϵ_F the Fermi energy. Dashed line: mean-field critical temperature T^*; solid line: Berezinskii-Kosterlitz-Thouless critical temperature T_{BKT}.

Phase fluctuations and superfluid fraction (I)

We now consider the effect of phase fluctuations, i.e. in Eq. (7) we allow $\theta(\mathbf{r},t) \neq 0$, but keep $\sigma(\mathbf{r},\tau) = 0$. To extract the contribution of the fluctuations we perform a gauge transformation, defining a new fermionic "neutral" field

$$\chi_s(\mathbf{r},\tau) = e^{i\theta(\mathbf{r},\tau)/2}\psi_s(\mathbf{r},\tau) \; . \tag{23}$$

In this way the Lagrangian density (6) becomes

$$\begin{aligned} \mathscr{L}_e &= \bar{\chi}_s \left[\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu\right]\chi_s + i\frac{\hbar^2}{2m}\bar{\chi}_s \boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}\chi_s \\ &+ \bar{\chi}_s\chi_s\left[-i\frac{\hbar}{2}\partial_\tau\theta - i\frac{\hbar^2}{4m}\nabla^2\theta + \frac{\hbar^2}{8m}(\nabla\theta)^2\right] \\ &+ \Delta_0\,\chi_\downarrow\,\chi_\uparrow + \Delta_0\bar{\chi}_\uparrow\,\bar{\chi}_\downarrow - \frac{\Delta_0^2}{g} \; . \end{aligned} \tag{24}$$

Phase fluctuations and superfluid fraction (II)

After functional integration over the new fermionic fields the partition function reads

$$Z = \int \mathcal{D}[\theta] \exp\left\{ -\frac{\tilde{S}_e(\theta)}{\hbar} \right\} \quad (25)$$

where

$$\frac{\tilde{S}_e(\theta)}{\hbar} = -Tr[\ln\left(G_0^{-1} + \Sigma_\theta\right)] - \beta L^2 \frac{\Delta_0^2}{g} \quad (26)$$

with G_0^{-1} given by Eq. (11) and Σ_θ given by

$$\begin{aligned} \Sigma_\theta &= \hat{I}\left(i\frac{\hbar^2}{4m}\nabla^2\theta + i\frac{\hbar^2}{2m}\boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla} \right) \\ &- \hat{\tau}_3\left(i\frac{\hbar}{2}\partial_\tau\theta - \frac{\hbar^2}{8m}(\nabla\theta)^2 \right) . \end{aligned} \quad (27)$$

Here $\hat{I}$ is the 2×2 identity matrix and $\hat{\tau}_3$ is the third Pauli matrix.

Phase fluctuations and superfluid fraction (III)

At the second order in a gradient expansion of Σ_θ the partition function eventually can be written as

$$\mathcal{Z} = \exp\left\{-\frac{S_{mf}}{\hbar}\right\} \int \mathcal{D}[\theta] \exp\left\{-\frac{S_\theta}{\hbar}\right\} , \tag{28}$$

where S_{mf} is given by Eq. (10), while the action functional S_θ of the phase is given by

$$S_\theta = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r} \left\{ \frac{J}{2} \left(\nabla\theta\right)^2 + \frac{K_{\theta\theta}}{2} \left(\partial_\tau\theta\right)^2 \right\} , \tag{29}$$

where

$$J = \frac{\hbar^2}{4mL^2} \sum_{\mathbf{k}} \left[1 - \frac{\frac{\hbar^2 k^2}{2m} - \mu}{E_k} X_T(E_k) - \frac{\hbar^2 k^2}{2m} X_T'(E_k) \right] , \tag{30}$$

is the stiffness,

$$K_{\theta\theta} = \frac{\hbar^2}{4L^2} \sum_{\mathbf{k}} \left[\frac{\Delta_0^2}{E_k^3} X_T(E_k) + \frac{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2}{E_k^2} X_T'(E_k) \right] . \tag{31}$$

is the phase susceptibility, and $X_T(E_k) = \tanh\left(\beta E_k/2\right)$.

Phase fluctuations and superfluid fraction (III)

The action functional (29) has the form of a 2D quantum XY model, where the Goldstone field $\theta(\mathbf{r},\tau)$ is defined in principle as an angular variable.
The temperature T_{BKT} of the Berezinskii-Kosterlitz-Thouless superfluid-normal phase transition can be estimated by solving self-consistently:

$$k_B\, T_{BKT} = \frac{\pi}{2} J(T_{BKT}) \,, \tag{32}$$

where $J(T)$ is defined by Eq. (30) with μ and Δ_0 given by the solutions of the gap and number equations Eqs. (16) and(18).

Phase fluctuations and superfluid fraction (IV)

Since $\mathbf{v}_s = (\hbar/m)\boldsymbol{\nabla}\theta$ is the superfluid velocity, the term $(J/2)(\nabla\theta)^2$ may be identified with the superfluid kinetic energy density $(1/2)n_s v_s^2$, where

$$n_s = \frac{4m}{\hbar^2} J \tag{33}$$

is the superfluid number density. The renormalization group theory dictates that for a 2D uniform system above T_{BKT} the phase stiffness J, and consequently also superfluid density n_s, is strictly zero.

Phase fluctuations and superfluid fraction (V)

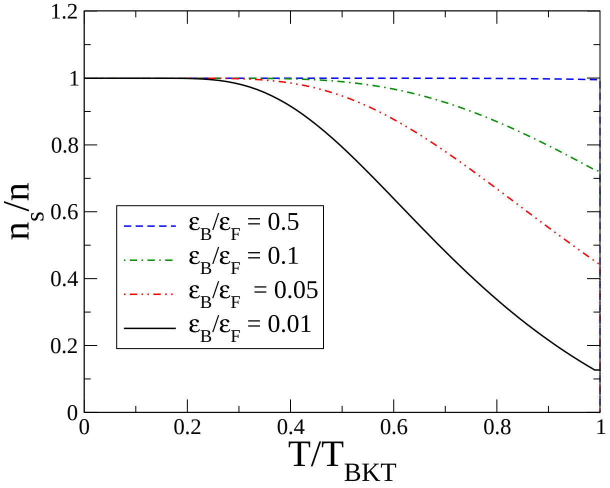

Figure: Superfluid fraction n_s/n as a function of the scaled temperature T/T_{BKT} for different values of the scaled binding energy ϵ_B/ϵ_F, where $\epsilon_F = (\hbar^2/m)\pi n$ is the Fermi energy.

Phase and amplitude fluctuations and sound velocity (I)

The velocity of the Goldstone mode is nothing else than the first sound velocity of the superfluid and it is given by

$$c_s = \sqrt{\frac{J}{K}} \,, \tag{34}$$

where J is the stiffness and K is the susceptibility.
Within the phase-only approach we have $K = K_{\theta\theta}$, and using Eqs. (30) and (31) at zero temperature one immediately finds

$$J = \frac{\epsilon_F}{4\pi} \,, \tag{35}$$

and

$$K_{\theta\theta} = \frac{m}{4\pi} \frac{\epsilon_F}{\epsilon_F + \frac{1}{2}\epsilon_B} \,, \tag{36}$$

and consequently, using Eq. (34) with $K = K_{\theta\theta}$, we obtain

$$c_s = \frac{v_F}{\sqrt{2}} \sqrt{1 + \frac{1}{2}\frac{\epsilon_B}{\epsilon_F}} \qquad \text{at } T = 0 \text{ (phase-only)} \,, \tag{37}$$

where $v_F = \sqrt{2\epsilon_F/m}$ is the Fermi velocity and $\epsilon_F = (\hbar^2/m)\pi n$ is the Fermi energy.

Phase and amplitude fluctuations and sound velocity (II)

We now consider both phase $\theta(\mathbf{r},\tau)$ and amplitude $\sigma(\mathbf{r},\tau)$ fluctuations in $\Delta(\mathbf{r},\tau)$.
In our zero-temperature 2D system after integration over $\sigma(\mathbf{r},\tau)$ we obtain the action functional S_θ of Eq. (29) with the stiffness J still given by Eq. (30) but with a new K instead of $K_{\theta\theta}$. In particular, the new susceptibility K is given by

$$K = \frac{K_{\theta\theta}K_{\sigma\sigma} - K_{\sigma\theta}^2}{K_{\sigma\sigma}} , \qquad (38)$$

which is a non trivial combination of the phase-only susceptibility $K_{\theta\theta}$ given by Eq. (31), the amplitude-only susceptibility $K_{\sigma\sigma}$ and the amplitude-phase susceptibility $K_{\sigma\theta}$. Note that only when amplitude and phase fluctuations are decoupled, i.e. when $K_{\sigma\theta} \simeq 0$ one obtains $K \simeq K_{\theta\theta}$.

Phase and amplitude fluctuations and sound velocity (III)

At zero temperature we easily find that $K_{\theta\theta}$ is indeed given by Eq. (36), while $K_{\sigma\sigma}$ and $K_{\sigma\theta}$ are

$$K_{\sigma\sigma} = -\frac{m}{8\pi\epsilon_B}\frac{\Delta_0^2}{\epsilon_F + \frac{1}{2}\epsilon_B}, \quad (39)$$

$$K_{\sigma\theta} = \frac{m}{8\pi}\frac{\Delta_0}{\epsilon_F + \frac{1}{2}\epsilon_B}. \quad (40)$$

It follows that the sound velocity of the 2D superfluid system reads

$$c_s = \frac{v_F}{\sqrt{2}} \qquad \text{at } T = 0 \text{ (phase and amplitude)}, \quad (41)$$

Phase and amplitude fluctuations and sound velocity (IV)

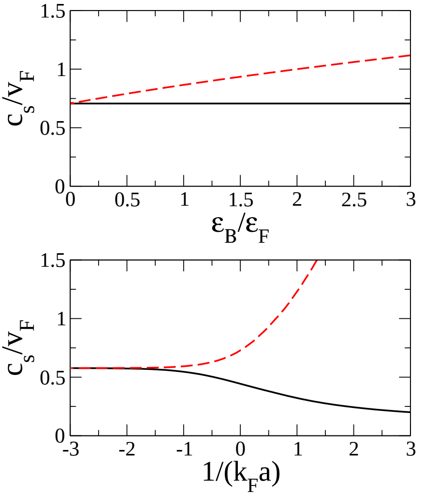

Figure: Sound velocity c_s at zero temperature ($T = 0$) with only phase fluctuations (dashed lines) and with phase and amplitude fluctuations (solid lines). Upper panel: 2D Fermi superfluid. Lower panel: 3D Fermi superfluid.

Open problems

There are several open problems regarding our 2D Fermi superfluid in the BCS-BEC crossover. Among them we mention:

- first and second sound at finite temperature
- condensate fraction with amplitude and phase fluctuations at $T = 0$
- beyond mean-field equation of state: comparison with MC results of Bertaina and Giorgini, PRL **106**, 110403 (2011)
- unbalanced system