

# Coleman-Weinberg quantum effective action for Josephson dynamics

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# Coleman-Weinberg quantum effective action

In the 1970s, in the context of **relativistic quantum field theory** for a scalar field  $\Phi(x)$ , it was shown by Coleman and Weinberg<sup>1</sup> and further formalized by Coleman, Jackiw, and Politzer<sup>2</sup> that the quantum effective action  $\Gamma$  admits the one-loop expansion

$$\Gamma[\Phi] = S[\Phi] + \frac{i\hbar}{2} \text{Tr} \left[ \ln \left( \frac{\delta^2 S}{\delta \Phi^2} [\Phi] \right) \right] \quad (1)$$

where the trace-log term encodes quantum fluctuations around the classical solution, with  $\text{Tr}[\dots]$  denoting the sum over the eigenmodes of the fluctuation operator. Eq. (1) gives quantum corrections to a classical field theory  $S[\Phi]$  at the Gaussian (one-loop) level.

We will apply the same idea to **collective dynamical variables** of **non-relativistic Josephson junctions**.

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<sup>1</sup>S. Coleman and S. Weinberg, Phys. Rev. D **7**, 1888 (1973).

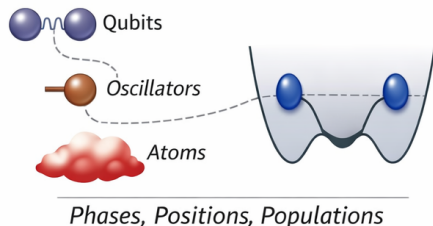
<sup>2</sup>S. Coleman, R. Jackiw, H. D. Politzer, Phys. Rev. D **10**, 2491 (1974).

# Bosonic Josephson tunneling (I)

A system of  $N$  interacting bosons confined by a symmetric double-well potential can be described by the two-site Bose-Hubbard model

$$\hat{H} = -J (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{U}{2} [\hat{N}_1(\hat{N}_1 - 1) + \hat{N}_2(\hat{N}_2 - 1)] \quad (2)$$

with  $J > 0$  the tunneling (hopping) energy,  $U$  the boson-boson interaction, and  $\hat{N}_j = \hat{a}_j^\dagger \hat{a}_j$ . Here  $\hat{a}_1$  and  $\hat{a}_j^\dagger$  are the bosonic ladder operators.



## Bosonic Josephson tunneling (II)

The mean-field approximation is obtained<sup>3</sup> by using Glauber coherent states

$$|\psi(t)\rangle = |\alpha_1(t)\rangle_1 |\alpha_2(t)\rangle_2 \quad (3)$$

where  $|\alpha_j(t)\rangle$  is the eigenstate of the annihilation operator  $\hat{a}_j$ , with complex eigenvalue

$$\alpha_j(t) = \sqrt{N_j(t)} e^{i\phi_j(t)}, \quad (4)$$

where  $N_j(t) = \langle \psi(t) | \hat{N}_j | \psi(t) \rangle$  is the average number of bosons in the site  $j = 1, 2$  and  $\phi_j(t)$  is the corresponding phase.

Quite remarkably, the mean-field dynamics is obtained by extremizing the following action functional

$$S = \int \langle \psi(t) | \left( i\hbar \frac{\partial}{\partial t} - \hat{H} \right) | \psi(t) \rangle. \quad (5)$$

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<sup>3</sup>R. Franzosi and V. Penna, Phys. Rev. E **67**, 046227 (2003).

# Bosonic Josephson tunneling (II)

One can also introduce<sup>4</sup> the relative phase

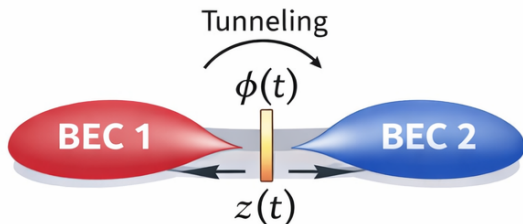
$$\phi(t) = \phi_2(t) - \phi_1(t) \quad (6)$$

and the normalized population imbalance

$$z(t) = \frac{N_1(t) - N_2(t)}{N} \in [-1, 1] \quad (7)$$

Here  $N = N_1(t) + N_2(t)$  is a constant of motion.

In this framework  $\phi(t)$  and  $z(t)$  are the time-dependent variational parameters of the coherent state  $|\psi(t)\rangle$  which extremizes the action  $S$ .



<sup>4</sup>A. Smerzi *et al.*, Phys. Rev. Lett. **79**, 4950 (1997).

# Bosonic Josephson tunneling (III)

Specifically, we find<sup>5</sup>

$$S[z, \phi] = \int dt \left[ \frac{N\hbar z}{2} \dot{\phi} - \frac{UN^2}{4} z^2 + JN\sqrt{1-z^2} \cos \phi \right], \quad (8)$$

with  $\phi(t)$  and  $z(t)$  Lagrangian variables.

The corresponding Euler-Lagrange equations are

$$\hbar \dot{\phi} = \frac{2Jz}{\sqrt{1-z^2}} \cos \phi + UNz, \quad (9a)$$

$$\hbar \dot{z} = -2J\sqrt{1-z^2} \sin \phi. \quad (9b)$$

Linearizing around  $z = 0$  and  $\phi = 0$  one gets the Josephson frequency

$$\omega_J = \sqrt{2J(UN + 2J)}/\hbar. \quad (10)$$

This prediction was experimentally verified in 2005 with <sup>87</sup>Rb atoms.<sup>6</sup>

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<sup>5</sup>S. Wimberger, G. Manganelli, A. Brollo, L.S., Phys. Rev. A **103**, 023326 (2021).

<sup>6</sup>M. Albiez *et al.*, Phys. Rev. Lett. **95**, 010402 (2005).

# Phase-only effective action

Given the action  $S[z, \phi]$ , the effective phase-only action  $S[\phi]$  is defined as<sup>7</sup>

$$e^{\frac{i}{\hbar} S[\phi]} = \int \mathcal{D}[z] e^{\frac{i}{\hbar} S[z, \phi]} . \quad (11)$$

The path integral can be computed explicitly expanding  $S[z, \phi, ]$  up to second order around  $z = 0$ . The resulting only-phase mean-field action is given by

$$S[\phi] = \int dt \left[ \frac{m(\phi)}{2} \dot{\phi}^2 - V(\phi) \right] , \quad (12)$$

where

$$m(\phi) = \frac{N\hbar^2}{2(UN + 2J \cos(\phi))} . \quad (13)$$

$$V(\phi) = -JN \cos(\phi) . \quad (14)$$

Quite remarkably, with the only-phase action  $S[\phi]$  one recovers exactly the same mean-field Josephson frequency obtained with  $S[z, \phi]$ .

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<sup>7</sup>K. Furutani, J. Tempere, L.S., Phys. Rev. B **105**, 134510 (2022).

# Phase-only quantum effective action (I)

The one-loop quantum effective action<sup>8</sup>

$$\Gamma[\phi] = S[\phi] + \frac{i\hbar}{2} \text{Tr} \ln \left( \frac{\delta^2 S}{\delta\phi^2}[\phi] \right) \quad (15)$$

provides a systematic way to include beyond-mean-field (quantum) fluctuations. At zero temperature we find<sup>9</sup>

$$\Gamma[\phi] = \int dt \left[ \frac{m_{\text{eff}}(\phi)}{2} \dot{\phi}^2 - V_{\text{eff}}(\phi) \right], \quad (16)$$

where

$$m_{\text{eff}}(\phi) = m(\phi) + \frac{\hbar}{32} \frac{(\partial_\phi \Omega(\phi)^2)^2}{\Omega(\phi)^5} \quad (17)$$

$$V_{\text{eff}}(\phi) = V(\phi) + \frac{\hbar \Omega(\phi)}{2} \quad (18)$$

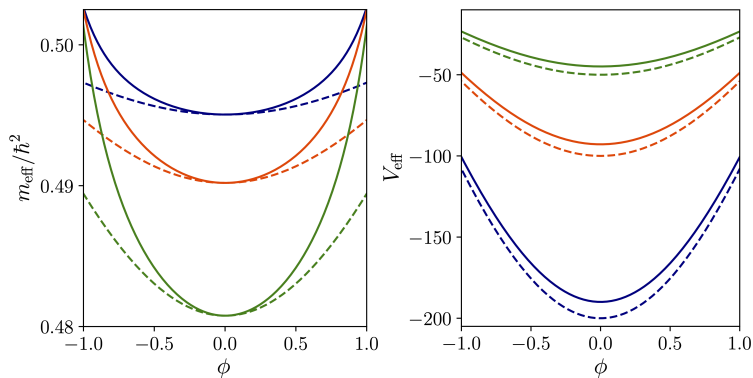
with

$$\Omega(\phi)^2 = \frac{V''(\phi) - \frac{m'(\phi)}{2m(\phi)} V'(\phi)}{m(\phi)}. \quad (19)$$

<sup>8</sup>S. Coleman, R. Jackiw, H.D. Politzer, Phys. Rev. D **10**, 2491 (1974).

<sup>9</sup>C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

# Phase-only quantum effective action (II)



Effective mass (left panel) and effective potential as functions of  $\phi$  for  $U = J = 1.0$  and  $N = 50$  (green lines), 100 (orange lines), and 200 (blue lines). The dashed lines represent the corresponding mean-field result. Adapted from C. Vianello, S. Salvatore, L.S., *Int. J. Theor. Phys.* **64**, 315 (2025).

# Phase-only quantum effective action (III)

Quantum corrections do not change the position of the minimum of the effective potential  $V_{\text{eff}}(\phi)$ , which is still located at  $\phi = 0$ , where also  $m'_{\text{eff}}(0) = 0$ . In particular, small oscillations around  $\phi = 0$  are harmonic, with the quantum-corrected frequency

$$\tilde{\omega}_J = \sqrt{\frac{V''_{\text{eff}}(0)}{m_{\text{eff}}(0)}} = \omega_J \sqrt{1 - \frac{1}{2N} \frac{UN + 6J}{\sqrt{2J(UN + 2J)}}}, \quad (20)$$

where

$$\omega_J = \frac{\sqrt{2J(UN + 2J)}}{\hbar} \quad (21)$$

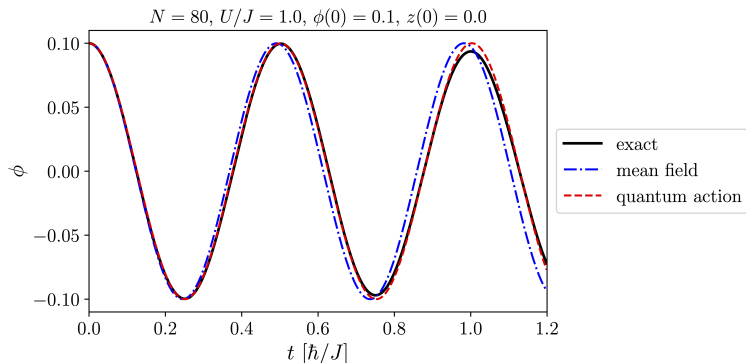
is the mean-field Josephson frequency.

- Exact numerical results<sup>10</sup> confirm the robustness of Eq. (20).
- The relative correction induced by quantum fluctuations can be of 3% for condensates with  $N = 100$  atoms in realistic trapping configurations.

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<sup>10</sup>C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

# Phase-only quantum effective action (IV)



Comparison between the exact dynamics (solid black line), the mean-field dynamics (dashed-dotted blue line), and the quantum-corrected dynamics (dashed red line) of the relative phase, for  $N = 80$ ,  $U = J = 1.0$ ,  $\phi(0) = 0.1$ , and  $\dot{\phi}(0) = 0$ . Adapted from C. Vianello, S. Salvatore, L.S., *Int. J. Theor. Phys.* **64**, 315 (2025).

# Dissipation (I)

Let us consider the **damped** mean-field equation

$$m\ddot{\phi} + \gamma\dot{\phi} + V'(\phi) = 0 \quad (22)$$

with  $\gamma > 0$  the **damping coefficient** and  $m = \hbar^2/(2U)$  the “mass” in the Josephson regime.

Due to the presence of dissipation, we do not invoke the least action principle but we instead use the Lagrange-d'Alembert principle

$$\delta S[\phi] + \delta S_\gamma[\phi] = 0, \quad (23)$$

where  $\delta S$  is the **exact 1-form** (first variation) with conservative action

$$S[\phi] = \int dt \left[ \frac{m}{2} \dot{\phi}^2 - V(\phi) \right], \quad (24)$$

while

$$\delta S_\gamma[\phi] = - \int dt \gamma \dot{\phi} \delta\phi \quad (25)$$

is a **non-exact 1-form** for dissipation, where  $\delta W_\gamma = -\gamma \dot{\phi} \delta\phi$  plays the role of infinitesimal virtual work.

## Dissipation (II)

Writing

$$\phi(t) = \bar{\phi}(t) + \eta(t) \quad (26)$$

and linearizing the damped equation of motion, one obtains

$$m\ddot{\eta} + \gamma\dot{\eta} + V''(\bar{\phi})\eta = 0. \quad (27)$$

The linearized dissipative dynamics defines the fluctuation operator

$$\hat{M}_\gamma = -m\partial_t^2 - \gamma\partial_t - V''(\bar{\phi}). \quad (28)$$

Gaussian fluctuations around  $\bar{\phi}(t)$  are described by the real-time partition function

$$\mathcal{Z}^{(2)}[\bar{\phi}] = \int \mathcal{D}[\eta] e^{\frac{i}{\hbar} \int dt \eta \hat{M}_\gamma \eta} = e^{\frac{i}{\hbar} Q_\gamma[\bar{\phi}]}, \quad (29)$$

where

$$Q_\gamma[\bar{\phi}] = \frac{i\hbar}{2} \text{Tr}[\ln(\hat{M}_\gamma)]. \quad (30)$$

# Dissipation (III)

Assuming that  $\bar{\phi}(t)$  varies slowly in time, one may perform a derivative expansion of  $\text{Tr}[\ln(\hat{M}_\gamma)]$ . To leading order one obtains<sup>11</sup>

$$Q_\gamma[\bar{\phi}] = \int dt \left[ -\frac{\hbar}{2} \Omega_\gamma(\bar{\phi}) \right], \quad (31)$$

with

$$\Omega_\gamma(\bar{\phi}) = \sqrt{\frac{V''(\bar{\phi})}{m} - \left(\frac{\gamma}{2m}\right)^2}. \quad (32)$$

Thus, the quantum effective action can be written as

$$\Gamma_\gamma[\bar{\phi}] = S[\bar{\phi}] + Q_\gamma[\bar{\phi}]. \quad (33)$$

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<sup>11</sup>C. Vianello, A. Bardin, and LS, e-preprint arXiv:2605.19643.

## Dissipation (IV)

The quantum-corrected equation of motion can be extracted<sup>12</sup> from

$$\delta\Gamma_\gamma[\bar{\phi}] + \delta\mathcal{S}_\gamma[\bar{\phi}] = 0. \quad (34)$$

Thus, the quantum-corrected dissipative dynamics reads

$$m\ddot{\bar{\phi}} + \gamma\dot{\bar{\phi}} + V'_{\text{eff}}(\bar{\phi}) = 0. \quad (35)$$

Quite remarkably, the structure of the effective potential is similar to the one of the conservative case:

$$V_{\text{eff}}(\bar{\phi}) = V(\bar{\phi}) + \frac{1}{2}\hbar\Omega_\gamma(\bar{\phi}) \quad (36)$$

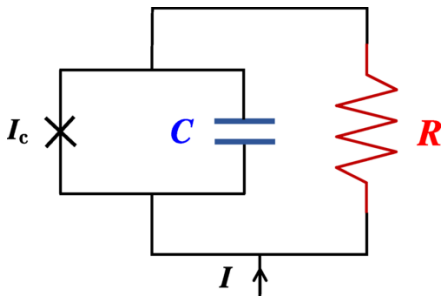
but now

$$\Omega_\gamma(\bar{\phi}) = \sqrt{\frac{V''(\bar{\phi})}{m} - \left(\frac{\gamma}{2m}\right)^2}. \quad (37)$$

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<sup>12</sup>For a systematic derivation within the **Schwinger–Keldysh** framework, see C. Vianello, A. Bardin, and C. LS, e-preprint arXiv:2605.19643.

# Dissipation (V)



Let us consider a resistively ( $R$ ) and capacitively ( $C$ ) shunted Josephson junction. The phase-only mean-field equation of motion is

$$\frac{\hbar^2 C}{4e^2} \ddot{\phi} + \frac{\hbar^2}{4e^2 R} \dot{\phi} + V'(\phi) = 0 \quad (38)$$

with washboard potential

$$V(\phi) = -\frac{\hbar}{2e} (I\phi + I_c \cos \phi) , \quad (39)$$

external electric current  $I$ , and critical current  $I_c = 2eJN/\hbar$ . Here  $2e$  is the electric charge of a Cooper pair.

For  $I = 0$ , small oscillations around  $\phi = 0$  our approach<sup>13</sup> gives

$$\tilde{\omega}_d = \sqrt{\omega_J^2 - \omega_{RC}^2} \sqrt{1 - \frac{e^2}{\hbar C \omega_J [1 - (\omega_{RC}/\omega_J)^2]^{3/2}}} \quad (40)$$

where

$$\omega_J = \sqrt{\frac{2eI_c}{\hbar C}}, \quad \omega_{RC} = \frac{1}{2RC}. \quad (41)$$

The frequency  $\tilde{\omega}_d$  is the frequency of the damped oscillation, that is modified by quantum fluctuations. The small-amplitude dynamics is therefore

$$\bar{\phi}(t) = \phi_0 e^{-\omega_{RC}t} \cos(\tilde{\omega}_d t + \delta), \quad (42)$$

with  $\phi_0$  and  $\delta$  fixed by the initial conditions.

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<sup>13</sup>C. Vianello, A. Bardin, and LS, e-preprint arXiv:2605.19643.

# Conclusions

- The quantum effective action method is useful not only in relativistic QFT but also in non-relativistic MQ.
- It provides a bridge between classical (or mean-field) dynamics and quantum fluctuations.
- It is relevant for Josephson junctions.
- One can include dissipation.
- One could also include thermal effects.

## Thank you for attention!

Thanks a lot also to my Josephson **collaborators**: Andrea Bardin (DFA UNIPD), Alberto Brollo (Tech. Univ. Munchen), Oliver Hideg (Bonn Univ.), Koichiro Furutani (Nagoya Univ.), Francesco Lorenzi (DEI UNIPD), Gabriele Manganelli (Cornell Univ.), Sofia Salvatore (UNIPD), Jacques Tempere (Antwerp Univ.), Cesare Vianello (DFA UNIPD), and Sandro Wimberger (Univ. Parma).

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## BACKUP SLIDES

# Backup: Keldysh-like quantum effective action (I)

We introduce an **auxiliary response field**  $\tilde{\phi}(t)$  and write the Martin–Siggia–Rose–Keldysh action

$$S_K[\phi, \tilde{\phi}] = \int dt [-m\ddot{\phi} - \gamma\dot{\phi} - V'(\phi)] \tilde{\phi} + \frac{i}{2} \int dt dt' \tilde{\phi}(t) \alpha_K(t-t') \tilde{\phi}(t'). \quad (43)$$

Here  $\alpha_K$  is the fluctuation (noise) kernel. The Keldysh action  $S_K$  is generally complex: the imaginary part encodes the fluctuations of the environment.

The fluctuation kernel satisfies the fluctuation-dissipation relation

$$\alpha_K(\omega) = \hbar \coth\left(\frac{\beta\hbar\omega}{2}\right) \text{Im} \alpha_R(\omega), \quad \text{Im} \alpha_R(\omega) = \gamma\omega, \quad (44)$$

therefore

$$\alpha_K(\omega) = \gamma\hbar\omega \coth\left(\frac{\beta\hbar\omega}{2}\right) \quad (45)$$

where  $\beta = 1/(k_B T)$  with  $k_B$  the Boltzmann constant and  $T$  the temperature of the bath.

## Backup: Keldysh-like quantum effective action (II)

The physical equation of motion follows from the stationarity condition with respect to the response field  $\tilde{\phi}$ , i.e.

$$\frac{\delta S_K}{\delta \tilde{\phi}} = 0, \quad (46)$$

which gives the Langevin equation

$$m\ddot{\phi} + \gamma\dot{\phi} + V'(\phi) = \xi(t), \quad \langle \xi(t)\xi(t') \rangle = \alpha_K(t-t'). \quad (47)$$

We expand around background fields

$$\phi = \bar{\phi} + \eta, \quad \tilde{\phi} = \bar{\tilde{\phi}} + \tilde{\eta}. \quad (48)$$

Here  $\bar{\phi}$  and  $\bar{\tilde{\phi}}$  are background fields, while  $\eta$  and  $\tilde{\eta}$  are Gaussian fluctuations. The second variation reads

$$\delta^2 S_K = \frac{1}{2} \int dt dt' \begin{pmatrix} \eta & \tilde{\eta} \end{pmatrix} \hat{F}_\gamma(t, t') \begin{pmatrix} \eta \\ \tilde{\eta} \end{pmatrix}, \quad (49)$$

# Backup: Keldysh-like quantum effective action (III)

with

$$\hat{F}_\gamma = \begin{pmatrix} -V''''(\bar{\phi})\bar{\phi} & \hat{M}_\gamma^\dagger \\ \hat{M}_\gamma & -i\alpha_\kappa \end{pmatrix}, \quad (50)$$

where

$$\hat{M}_\gamma = -m\partial_t^2 - \gamma\partial_t - V''(\bar{\phi}) \quad (51)$$

is the dissipative fluctuation operator.

Gaussian integration gives the Keldysh-like quantum effective action<sup>14</sup>

$$\Gamma_\kappa[\bar{\phi}, \bar{\phi}] = S_\kappa[\bar{\phi}, \bar{\phi}] + \frac{i\hbar}{2} \text{Tr}[\ln(\hat{F}_\gamma)]. \quad (52)$$

For low temperature and weak damping,

$$V_{\text{eff}}(\bar{\phi}) = V(\bar{\phi}) + \frac{\hbar}{2} \Omega_\gamma(\bar{\phi}), \quad (53)$$

with

$$\Omega_\gamma(\bar{\phi}) = \sqrt{\frac{V''(\bar{\phi})}{m} - \left(\frac{\gamma}{2m}\right)^2}. \quad (54)$$

<sup>14</sup>C. Vianello, A. Bardin, and C. LS, e-preprint arXiv:2605.19643.