Condensate fraction in ultracold Fermi atoms and in neutron matter

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Summary

- BCS-BEC equations and condensate fraction
- Results for 3D ultracold atoms
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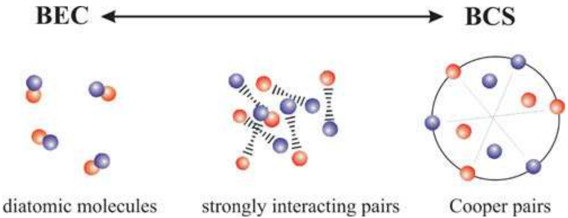
BCS-BEC equations and condensate fraction (I)

In 2004 the BCS-BEC crossover has been observed with **ultracold gases made of fermionic ^{40}K and ^{6}Li alkali-metal atoms.**[1]



This crossover is obtained by changing (with a Feshbach resonance) the s-wave scattering length a of the inter-atomic potential:

– $a \to 0^-$ (BCS regime of weakly-interacting Cooper pairs)

– $a \to \pm\infty$ (unitarity limit of strongly-interacting Cooper pairs)

– $a \to 0^+$ (BEC regime of bosonic dimers)

[1] C.A. Regal et al., PRL **92**, 040403 (2004); M.W. Zwierlein et al., PRL **92**, 120403 (2004); M. Bartenstein, A. Altmeyer et al., PRL **92**, 120401 (2004); J. Kinast et al., PRL **92**, 150402 (2004).

BCS-BEC equations and condensate fraction (II)

The crossover from a BCS superfluid ($a < 0$) to a BEC of molecular pairs ($a > 0$) has been investigated experimentally around a Feshbach resonance, where the s-wave scattering length a diveges, and it has been shown that the system is (meta)stable.
The detection of quantized vortices under rotation[2] has clarified that this dilute gas of ultracold atoms is superfluid.
Usually the BCS-BEC crossover is analyzed in terms of

$$y = \frac{1}{k_F a} \tag{1}$$

the inverse scaled interaction strength, where $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wave number and n the total density.
The system is dilute because $r_e k_F \ll 1$, with r_e the effective range of the inter-atomic potential.

[2] M.W. Zwierlein *et al.*, Science **311**, 492 (2006); M.W. Zwierlein *et al.*, Nature **442**, 54 (2006)

BCS-BEC equations and condensate fraction (III)

The shifted Hamiltonian of the **uniform two-spin-component Fermi superfluid** is given by

$$\hat{H}' = \int d^3\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^+_\sigma(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu \right) \hat{\psi}_\sigma(\mathbf{r}) + g\, \hat{\psi}^+_\uparrow(\mathbf{r})\, \hat{\psi}^+_\downarrow(\mathbf{r})\, \hat{\psi}_\downarrow(\mathbf{r})\, \hat{\psi}_\uparrow(\mathbf{r}) \,, \tag{2}$$

where $\hat{\psi}_\sigma(\mathbf{r})$ is the field operator that annihilates a fermion of spin σ in the position $\mathbf{r}$, while $\hat{\psi}^+_\sigma(\mathbf{r})$ creates a fermion of spin σ in $\mathbf{r}$. Here $g < 0$ is the strength of the attractive fermion-fermion interaction.

BCS-BEC equations and condensate fraction (IV)

The ground-state average of the number of fermions reads

$$N = \int d^3\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \langle \hat{\psi}^+_\sigma(\mathbf{r}) \, \hat{\psi}_\sigma(\mathbf{r}) \rangle \; . \tag{3}$$

This total number N is fixed by the chemical potential μ which appears in Eq. (2).
In a Fermi system the largest eigenvalue N_0 of the two-body density matrix gives the number of fermion pairs which have their center of mass with zero linear momentum. This condensed number of pairs is given by

$$N_0 = 2 \int d^3\mathbf{r}_1 \, d^3\mathbf{r}_2 \, |\langle \hat{\psi}_\downarrow(\mathbf{r}_1) \, \hat{\psi}_\uparrow(\mathbf{r}_2) \rangle|^2 \; . \tag{4}$$

BCS-BEC equations and condensate fraction (V)

Within the Bogoliubov approach the shifted Hamiltonian (2) can be diagonalized by using the Bogoliubov-Valatin representation of the field operator $\hat{\psi}_\sigma(\mathbf{r})$ in terms of the anticommuting quasi-particle Bogoliubov operators $\hat{b}_{\mathbf{k}\sigma}$ with amplitudes $u_\mathbf{k}$ and $v_\mathbf{k}$ and the quasi-particle energy $E_\mathbf{k}$. In this way one finds familiar expressions for these quantities:

$$E_\mathbf{k} = \left[(\epsilon_\mathbf{k} - \mu)^2 + \Delta^2 \right]^{1/2} \tag{5}$$

and

$$\begin{aligned} u_\mathbf{k}^2 &= \left(1 + (\epsilon_\mathbf{k} - \mu)/E_\mathbf{k}\right)/2 \qquad (6) \\ v_\mathbf{k}^2 &= \left(1 - (\epsilon_\mathbf{k} - \mu)/E_\mathbf{k}\right)/2 \, , \qquad (7) \end{aligned}$$

where $\epsilon_\mathbf{k} = \hbar^2 k^2/(2m)$ is the single-particle energy.

BCS-BEC equations and condensate fraction (VI)

The parameter Δ is the pairing gap, which satisfies the gap equation

$$-\frac{1}{g} = \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}}} \,, \tag{8}$$

where Ω is the volume of the uniform system. Notice that this equation is ultraviolet divergent and it must be regularized.
The equation for the total density $n = N/\Omega$ of fermions is obtained from Eq. (3) as

$$n = \frac{2}{\Omega} \sum_{\mathbf{k}} v_{\mathbf{k}}^2 \,. \tag{9}$$

Finally, from Eq. (4) one finds that the condensate density $n_0 = N_0/\Omega$ of paired fermions is given by[3]

$$n_0 = \frac{2}{\Omega} \sum_{\mathbf{k}} u_{\mathbf{k}}^2 v_{\mathbf{k}}^2 \,. \tag{10}$$

[3] L.S., N. Manini, A. Parola, PRA **72**, 023621 (2005).

Results for 3D ultracold atoms (I)

In three dimensions, a suitable regularization[4] of the gap equation is obtained by introducing the **inter-atomic scattering length** a via the equation

$$-\frac{1}{g} = -\frac{m}{4\pi\hbar^2 a} + \frac{1}{\Omega}\sum_{\mathbf{k}} \frac{m}{\hbar^2 k^2}\,, \tag{11}$$

and then subtracting this equation from the gap equation (8). In this way one obtains the three-dimensional **regularized gap equation**

$$-\frac{m}{4\pi\hbar^2 a} = \frac{1}{\Omega}\sum_{\mathbf{k}} \left(\frac{1}{2E_k} - \frac{m}{\hbar^2 k^2} \right), \tag{12}$$

which can be used to study the full BCS-BEC crossover[5] by changing the amplitude and sign of the s-wave scattering length a.

[4] Marini, Pistolesi, and Strinati, Eur. Phys. J. B **1**, 151 (1998).

[5] D.M. Eagles, PR **186**, 456 (1969); A.J. Leggett, in *Modern Trends in the Theory of Condensed Matter*, p. 13, edited by A. Pekalski and J. Przystawa (Springer, Berlin, 1980).

Results for 3D ultracold atoms (II)

Taking into account the functional dependence of the amplitudes u_k and v_k on μ and Δ, one finds[6] the condensate density

$$n_0 = \frac{m^{3/2}}{8\pi\hbar^3}\,\Delta^{3/2}\sqrt{\frac{\mu}{\Delta}+\sqrt{1+\frac{\mu^2}{\Delta^2}}}\,. \tag{13}$$

By the same techniques, also the two BCS-BEC equations can be written in a more compact form as

$$-\frac{1}{a} = \frac{2(2m)^{1/2}}{\pi\hbar^3}\,\Delta^{1/2}\,I_1\left(\frac{\mu}{\Delta}\right)\,, \tag{14}$$

$$n = \frac{(2m)^{3/2}}{2\pi^2\hbar^3}\,\Delta^{3/2}\,I_2\left(\frac{\mu}{\Delta}\right)\,, \tag{15}$$

where $I_1(x)$ and $I_2(x)$ are two monotonic functions which can be expressed in terms of elliptic integrals[7].

[6] L.S., N. Manini, A. Parola, PRA **72**, 023621 (2005).
[7] Marini, Pistolesi, and Strinati, Eur. Phys. J. B **1**, 151 (1998).

Results for 3D ultracold atoms (III)

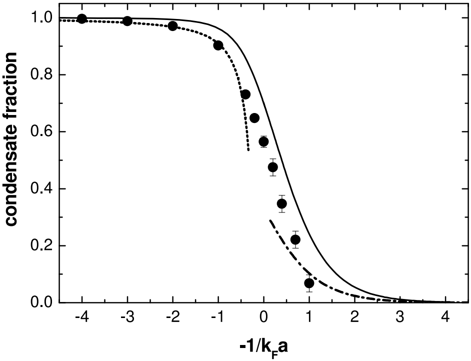


Figure: Condensate fraction of pairs as a function of the inverse interaction strength $y = 1/(k_F a)$: our mean-field theory (solid line); Fixed-Node Diffusion Monte Carlo results (symbols) [G. E. Astrakharchik et al., PRL **95**, 230405 (2005)]; Bogoliubov quantum depletion of a Bose gas with $a_m = 0.6a$ (dashed line); BCS theory (dot-dashed line).

Results for 2D ultracold atoms (I)

Contrary to the three-dimensional case, in two dimensions quite generally a **bound-state energy** ϵ_B exists for any value of the interaction strength g between atoms. For the contact potential the bound-state equation is

$$-\frac{1}{g} = \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 k^2}{2m} + \epsilon_B} , \tag{16}$$

and then subtracting this equation from the gap equation (8) one obtains the **two-dimensional regularized gap equation**[8]

$$\sum_{\mathbf{k}} \left(\frac{1}{\frac{\hbar^2 k^2}{2m} + \epsilon_B} - \frac{1}{2E_k} \right) = 0 . \tag{17}$$

[8] Marini, Pistolesi, and Strinati, Eur. Phys. J. B **1**, 151 (1998); M. Wouters, J. Tempere, J.T. Devreese, PRA **70**, 013616 (2004).

Results for 2D ultracold atoms (II)

In 2D we obtain[9] a remarkably simple formula for the condensed fraction

$$\frac{n_0}{n} = \frac{1}{4} \frac{\frac{\pi}{2} + \arctan\left(\frac{\mu}{\Delta}\right)}{\frac{\mu}{\Delta} + \sqrt{1 + \frac{\mu^2}{\Delta^2}}} . \tag{18}$$

Nicely, in this equation the condensate fraction depends only on the parameter $x_0 = \mu/\Delta$. Indeed, all quantities scaled in units of the Fermi energy ϵ_F depend only on the parameter x_0. In particular, one finds

$$\frac{\epsilon_B}{\epsilon_F} = 2 \frac{\sqrt{1 + x_0^2} - x_0}{\sqrt{1 + x_0^2} + x_0} , \tag{19}$$

$$\frac{\Delta}{\epsilon_F} = 2 \left(\sqrt{1 + x_0^2} - x_0 \right) , \tag{20}$$

and also

$$\frac{\mu}{\epsilon_F} = 2x_0 \left(\sqrt{1 + x_0^2} - x_0 \right) . \tag{21}$$

[9] L.S, PRA **76**, 015601 (2007).

Results for 2D ultracold atoms (III)

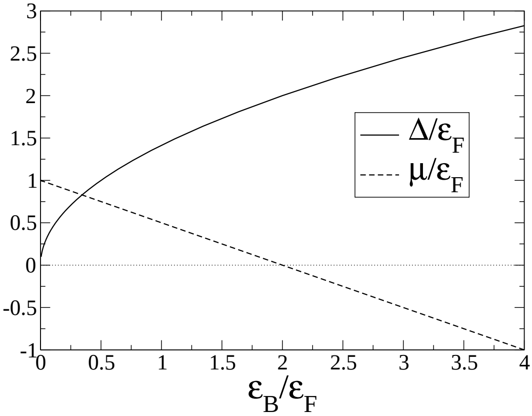


Figure: Energy gap Δ (solid line) and chemical potential μ (dashed line) in the uniform two-component dilute 2D Fermi gas as a function of scaled bound-state energy ϵ_B/ϵ_F. The horizontal dotted line simply shows the zero.

Results for 2D ultracold atoms (IV)

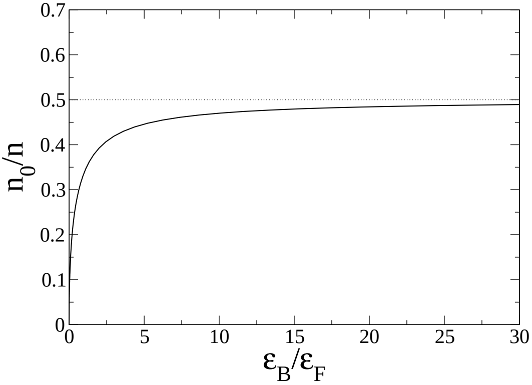


Figure: Condensate fraction n_0/n of Fermi pairs (solid line) in the uniform two-component dilute 2D Fermi gas as a function of scaled bound-state energy ϵ_B/ϵ_F. The horizontal dotted line shows the asymptotic value $n_0/n = 1/2$.

Neutron matter (I)

The **neutron matter** is a dense Fermi liquid made of two-component (spin up and down) neutrons. The shifted Hamiltonian of the uniform neutron matter can be written as

$$\hat{H}' = \int d^3\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}_\sigma^+(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu \right) \hat{\psi}_\sigma(\mathbf{r}) + \int d^3\mathbf{r}\, d^3\mathbf{r}'\, \hat{\psi}_\uparrow^+(\mathbf{r})\, \hat{\psi}_\downarrow^+(\mathbf{r}')\, V(\mathbf{r}-\mathbf{r}')\, \hat{\psi}_\downarrow(\mathbf{r}')\, \hat{\psi}_\uparrow(\mathbf{r}) \,, \tag{22}$$

where $\hat{\psi}_\sigma(\mathbf{r})$ is the field operator that annihilates a neutron of spin σ in the position $\mathbf{r}$, while $\hat{\psi}_\sigma^+(\mathbf{r})$ creates a neutron of spin σ in $\mathbf{r}$. Here $V(\mathbf{r}-\mathbf{r}')$ is the nutron-neutron potential characterized by s-wave scattering length $a = -18.5$ fm and effective range $r_e = 2.7$ fm.

Neutron Matter (II)

One can apply the familiar Bogoliubov approach to diagonalize the effective quadratic Hamiltonian, but now the paring gap $\Delta_{\mathbf{k}}$ depends explcitly on the wave number $\mathbf{k}$ and satisfies the integral equation

$$\Delta_{\mathbf{q}} = \sum_{\mathbf{k}} V_{\mathbf{qk}} \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}} , \tag{23}$$

where

$$V_{\mathbf{qk}} = \langle \mathbf{q}, -\mathbf{q} | V | \mathbf{k}, -\mathbf{k} \rangle \tag{24}$$

is the wave-number representation of the neutron-neutron potential, and

$$E_{\mathbf{k}} = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu \right)^2 + |\Delta_{\mathbf{k}}|^2} . \tag{25}$$

Neutron matter (III)

Under the simplifying assumptions

$$\mu \simeq \epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \, , \qquad \Delta_{\mathbf{k}} \simeq \Delta \, , \tag{26}$$

in the continuum limit the gap equation of the neutron matter becomes

$$1 = \frac{1}{2} \int \frac{d^3\mathbf{k} \; d^3\mathbf{r}}{(2\pi)^3} \frac{V(\mathbf{r}) \; e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{(\frac{\hbar^2 k^2}{2m} - \epsilon_F) + \Delta^2}} \, . \tag{27}$$

Moreover, the number equation reads

$$n = \frac{1}{2} \frac{(2m)^{3/2}}{2\pi^2 \hbar^3} \, \Delta^{3/2} \, I_2\left(\frac{\epsilon_F}{\Delta}\right) \, , \tag{28}$$

where $I_2(x)$ is the monotonic function

$$I_2(x) = \int_0^{+\infty} y^2 \left(1 - \frac{y^2 - x}{\sqrt{(y^2 - x)^2 + 1}}\right) dy \, . \tag{29}$$

Neutron matter (IV)

In a similar way one gets the condensate density of neutron-neutron pairs

$$n_0 = \frac{m^{3/2}}{8\pi\hbar^3}\,\Delta^{3/2}\sqrt{\frac{\epsilon_F}{\Delta} + \sqrt{1 + \frac{\epsilon_F^2}{\Delta^2}}}\,. \tag{30}$$

These equations show that knowing the scaled energy gap Δ/ϵ_F one can determine the condensate fraction

$$\frac{n_0}{n} = \frac{\pi}{2^{5/2}}\frac{\sqrt{\frac{\epsilon_F}{\Delta} + \sqrt{1 + \frac{\epsilon_F^2}{\Delta^2}}}}{I_2(\frac{\epsilon_F}{\Delta})} \tag{31}$$

Notice that in the deep BCS regime where $\Delta/\epsilon_F \ll 1$ one finds

$$\frac{n_0}{n} = \frac{3\pi}{8}\frac{\Delta}{\epsilon_F}\,. \tag{32}$$

Neutron matter (V)

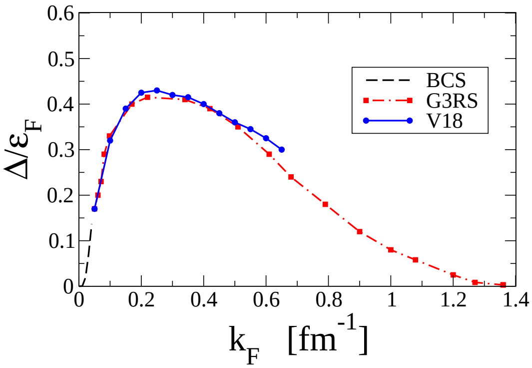


Figure: Scaled pairing gap Δ/ϵ_F vs Fermi wave number k_F. Dashed line: BCS limit; filled squares: results obtained with the G3RS nuclear potential [M. Matsuo, PRC **73**, 044309 (2006)]; filled circles: results obtained with the Argone V18 nuclear potential [A. Gezerlis and J. Carlson, PRC **81**, 025803 (2010)].

Neutron matter (VI)

Fitting the Matsuo data[10] of Δ/ϵ_F vs k_F we obtain the formula[11]

$$\frac{\Delta}{\epsilon_F} = \frac{\beta_0 k_F^{\beta_1}}{\exp(k_F^{\beta_2}/\beta_3) - \beta_3} \tag{33}$$

with the following fitting parameters:
$\beta_0 = 2.851$, $\beta_1 = 1.942$, $\beta_2 = 1.672$, $\beta_3 = 0.276$, $\beta_4 = 0.975$.
By using this fitting formula and the simple equation

$$\frac{n_0}{n} = \frac{\pi}{2^{5/2}} \frac{\sqrt{\frac{\epsilon_F}{\Delta} + \sqrt{1 + \frac{\epsilon_F^2}{\Delta^2}}}}{I_2(\frac{\epsilon_F}{\Delta})} \tag{34}$$

we get the condensate fraction of neutron matter as a function of the neutron density n.

[10] M. Matsuo, PRC **73**, 044309 (2006).
[11] L.S., PRC **84**, 067301 (2011)

Neutron matter (VII)

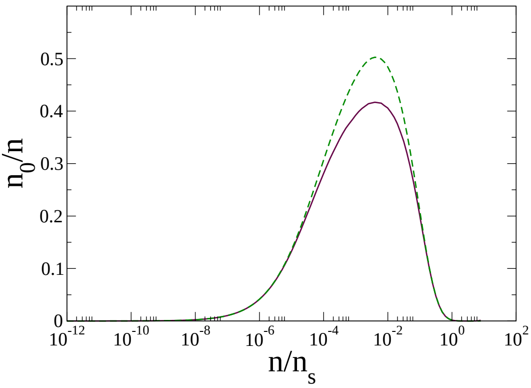
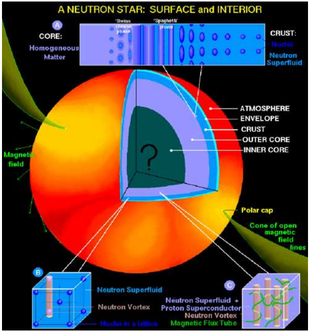


Figure: Condensate fraction n_0/n of neutron matter as a function of the scaled density n/n_s, where $n_s = 0.16$ fm^{-3} is the nuclear saturation density. The solid line is obtained by using Eqs. (31) and (33). The dashed line is obtained by using Eqs. (32) and (33).

Neutron stars (I)

Neutron stars are astronomical compact objects that can result from the gravitational collapse of a massive star during a supernova event. Such stars are mainly composed of neutrons.



Neutron stars are very hot and are supported against further collapse by Fermi pressure. A typical neutron star has a mass M between 1.35 and about 2.0 solar masses with a corresponding radius R of about 12 km.

Neutron stars (II)

Notice that in the crust of neutron stars one estimates[12] $T \simeq 10^8$ K, while $T_c \simeq 10^{10}$ K. Thus the crust of neutron stars is superfluid.
In previous slides we have found a fitting formula for the condensate fraction n_0/n of neutron matter as a function of the Fermi wave number

$$k_F = (3\pi^2 n)^{1/3} \; . \tag{35}$$

Knowing the density profile $n(r)$ of a neutron star[13], i.e. the neutron density n as a function of the distance r from the center of a neutron star, we can determine[14] the condensate fraction n_0/n of the neutron star as a function of the distance r.

[12] S. Zane, R. Turolla, and D. Page, Isolated Neutron Stars: from the Surface to the Interior (Springer, Berlin, 2007).

[13] B. Datta, A.V. Thampan, and D. Bhattacharya, J. Astrophys. Astr. **16**, 375 (1995).

[14] LS, in preparation

Neutron stars (III)

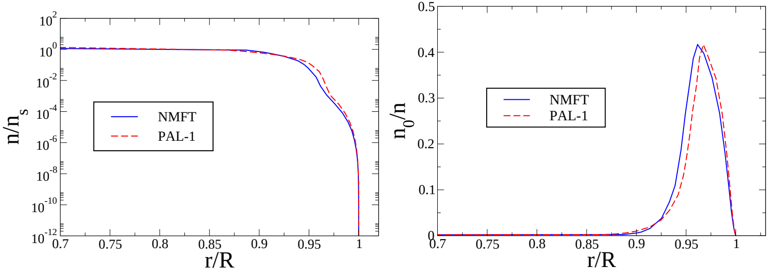


Figure: 1.4 solar mass neutron star. Left panel: Scaled density profile n/n_s vs scaled distance r/R. $n_s = 0.16\ \text{fm}^{-3}$ is the nuclear saturation density and R is the radius of the star. Right panel: condensate fraction n_0/n vs scaled distance r/R. Solid line is a simple neutron matter model [J.D. Walecka, Ann. Phys. **83**, 491 (1974)]. Dashed line is a more realistic model [T.L. Ainsworth, and J.M. Lattimer, PRL **61**, 2518 (1988)].

Conclusions

We have seen that the condensate fraction of Cooper pairs can be calculated in various superfluid fermionic systems: dilute atomic gases, dense neutron matter and neutron stars.

Our results on these and similar topics are published in

L.S., N. Manini, and A. Parola, PRA **72**, 023621 (2005).
L.S, PRA **76**, 015601 (2007).
L.S., PRA **83**, 033630 (2011).
L. Dell'Anna, G. Mazzarella, and L.S., PRA **84**, 033633 (2011).
L.S., PRC **84**, 067301 (2011).
L. Dell'Anna, G. Mazzarella, and L.S., PRA **86**, 053632 (2012).
L.S., PRA **86**, 055602 (2012).
L.S. and F. Toigo, PRA **86**, 023619 (2012).

Acknowledgments

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