Quantum effective action method: Josephson junctions

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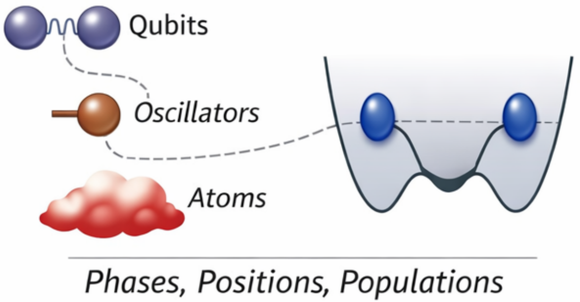
Motivation

- Collective variables: phases, positions, populations
- Mean-field often misses quantum fluctuations
- Exact many-body simulation is impractical for large systems
- Quantum Effective Action provides a systematic approach[1]



[1] S. Coleman, R. Jackiw, H. D. Politzer, Phys. Rev. D **10**, 2491 (1974); J. Iliopoulos, C. Itzykson, A. Martin, Rev. Mod. Phys. **47**, 165 (1975).

Outline

- Introduction to the method
- Technical derivation: one-loop effective action
- Case study: Josephson dynamics in BEC
- Conclusions

One-loop quantum effective action

In the 1970s, in the context of relativistic quantum field theory for a scalar field $\Phi(\mathbf{r}, t)$, it was proved[2] this remarkable one-loop expansion:

$$\Gamma[\Phi] = S[\Phi] - \frac{i\hbar}{2} \mathrm{Tr} \ln \left(\frac{\delta^2 S}{\delta \eta^2} [\Phi] \right) \tag{1}$$

- Γ: quantum effective action
- S: classical action
- $\eta(\mathbf{r}, t)$: fluctuation field around saddle-point solution

Eq. (1) gives quantum corrections to a classical field theory up to one-loop (Gaussian) fluctuations.

[2] S. Coleman, R. Jackiw, H. D. Politzer, Phys. Rev. D **10**, 2491 (1974); J. Iliopoulos, C. Itzykson, A. Martin, Rev. Mod. Phys. **47**, 165 (1975).

Non-relativistic effective quantum potential

In simple cases the field $\Phi(\mathbf{r}, t)$ can be a collective dynamical variable $q(t)$:

$$\Phi(\mathbf{r}, t) = q(t) \ . \tag{2}$$

The non-relativistic classical action could be

$$S[q] = \int dt \left[\frac{m}{2} \dot{q}^2 - V(q) \right] \tag{3}$$

and the corresponding quantum effective potential reads[3]

$$V_{\rm eff}(q) = V(q) + \frac{\hbar}{2} \sqrt{\frac{V''(q)}{m}} + k_B T \ln \left[1 - e^{-\hbar \sqrt{V''(q)/m}/(k_B T)} \right] \ . \tag{4}$$

- First term: classical potential
- Second term: quantum zero-point energy
- Third term: thermal fluctuations at temperature T

[3] L.S., Atoms **13**, 95 (2025).

Effective quantum potential for oscillations

Let us consider small oscillations around some equilibrium q^* of $V(q)$. In classical mechanics the corresponding oscillation frequency is

$$\omega = \sqrt{\frac{1}{m} V''(q^*)} \, . \tag{5}$$

Instead, by using the quantum effective potential we get

$$\omega_{\rm eff} = \sqrt{\frac{1}{m} V''_{\rm eff}(q^*)} \tag{6}$$

which gives quantum and thermal corrections to ω:

$$\omega_{\rm eff} = \omega_{\rm eff}(\hbar, T) \tag{7}$$

such that

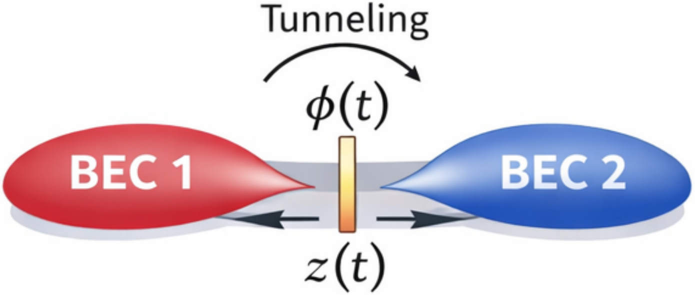
$$\omega_{\rm eff}(\hbar, T) \to \omega \quad \text{for } \hbar \to 0 \text{ and } T \to 0 \, . \tag{8}$$

Case study: Josephson tunneling with BEC (I)

- Variables: relative phase $\phi(t)$, population imbalance $z(t)$
- Mean-field (classical) tunneling action for N bosons:

$$S[\phi, z] = \int dt \left[\frac{\hbar N z}{2} \dot{\phi} - H(z, \phi) \right] \tag{9}$$

- Integrating out[4] z one gets the only-phase action for ϕ



[4] K. Furutani, J. Tempere, L.S., Phys. Rev. B **105**, 134510 (2022); C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

Case study: Josephson tunneling with BEC (II)

A system of N interacting bosons confined by an asymmetric double-well potential can be described by the two-site Bose-Hubbard model

$$\hat{H} = -J\left(\hat{a}_1^+\hat{a}_2 + \hat{a}_2^+\hat{a}_1\right) + \frac{U}{2}[\hat{N}_1(\hat{N}_1 - 1) + \hat{N}_2(\hat{N}_2 - 1)] \qquad (10)$$

with $J > 0$ the tunneling (hopping) energy, U the boson-boson interaction, and $\hat{N}_j = \hat{a}_j^+\hat{a}_j$. Here $\hat{a}_1$ and $\hat{a}_j^+$ are the bosonic ladder operators.

The mean-field approximation is obtained by using Glauber coherent states

$$|\psi(t)\rangle = |\alpha_1(t)\rangle_1\,|\alpha_2(t)\rangle_2 \qquad (11)$$

where $|\alpha_j(t)\rangle$ is the eigenstate of the annihilation operator $\hat{a}_j$, with complex eigenvalue

$$\alpha_j(t) = \sqrt{N_j(t)}\,e^{i\phi_j(t)}\ , \qquad (12)$$

where $N_j(t) = \langle\psi(t)|\hat{N}_j|\psi(t)\rangle$ is the average number of bosons in the site $j = 1, 2$ and $\phi_j(t)$ is the corresponding phase.

Case study: Josephson tunneling with BEC (III)

One can also introduce[5] the relative phase

$$\phi(t) = \phi_2(t) - \phi_1(t) \tag{13}$$

and the normalized population imbalance

$$z(t) = \frac{N_1(t) - N_2(t)}{N} \qquad \in [-1, 1] \tag{14}$$

Here $N = N_1(t) + N_2(t)$ is a constant of motion.
Quite remarkably, the mean-field dinamics is obtained by extremizing the following action functional

$$S[z, \phi] = \int \langle \psi(t) | \left(i\hbar \frac{\partial}{\partial t} - \hat{H} \right) | \psi(t) \rangle . \tag{15}$$

[5] A. Smerzi, S. Fantoni, S. Giovanazzi, S.R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997)

Case study: Josephson tunneling with BEC (IV)

Specifically, we find[6]

$$S[z,\phi] = \int dt \left[\frac{N\hbar z}{2}\dot{\phi} - \frac{UN^2}{4} z^2 + JN\sqrt{1-z^2}\cos\phi \right] \,, \qquad (16)$$

with $\phi(t)$ and $z(t)$ Lagrangian variables. Actually, for this specific problem $z(t)$ and $\phi(t)$ are canonically conjugated.
The corresponding Euler-Lagrange equations are

$$\hbar\dot{\phi} = \frac{2Jz}{\sqrt{1-z^2}}\cos\phi + UNz + \varepsilon, \qquad (17a)$$

$$\hbar\dot{z} = -2J\sqrt{1-z^2}\sin\phi \,. \qquad (17b)$$

Linearizing Eqs. (17a) and (17b) around $z = 0$ and $\phi = 0$ one gets the mean-field Josephson frequency

$$\omega_J = \frac{\sqrt{2J(UN + 2J)}}{\hbar} \,. \qquad (18)$$

[6] S. Wimberger, G. Manganelli, A. Brollo, L.S., Phys. Rev. A **103**, 023326 (2021).

Only-phase effective action

Given the action $S[z, \phi]$, the effective action for the phase $S[\phi]$ is defined as[7]

$$e^{\frac{i}{\hbar} S[\phi]} = \int \mathcal{D}[z] \, e^{\frac{i}{\hbar} S[z,\phi]} \, . \tag{19}$$

The path integral can be computed explicitly expanding $S[z, \phi,]$ up to second order around $z = 0$. The resulting only-phase mean-field action is given by

$$S[\phi] = \int dt \left[\frac{m(\phi)}{2} \dot{\phi}^2 - V(\phi) \right] \, , \tag{20}$$

where

$$m(\phi) = \frac{N\hbar^2}{2(UN + 2J\cos(\phi))} \, . \tag{21}$$

$$V(\phi) = -JN\cos(\phi) \, . \tag{22}$$

[7] K. Furutani, J. Tempere, L.S., Phys. Rev. B **105**, 134510 (2022).

Quantum corrections to Josephson dynamics (I)

The one-loop effective action:

$$\Gamma[\phi] = S[\phi] + \frac{i\hbar}{2} \mathrm{Tr} \ln \left(\frac{\delta^2 S}{\delta \eta^2} [\phi] \right) \qquad (23)$$

provides a systematic way to include beyond-mean-field (quantum) fluctuations. At zero temperature we find[8]

$$\Gamma[\phi] = \int dt \left[\frac{m_{\mathrm{eff}}(\phi)}{2} \dot{\phi}^2 - V_{\mathrm{eff}}(\phi) \right] , \qquad (24)$$

where

$$m_{\mathrm{eff}}(\phi) = m(\phi) + \frac{\hbar}{32} \frac{\left(\partial_\phi \Omega(\phi)^2\right)^2}{\Omega(\phi)^5} \qquad (25)$$

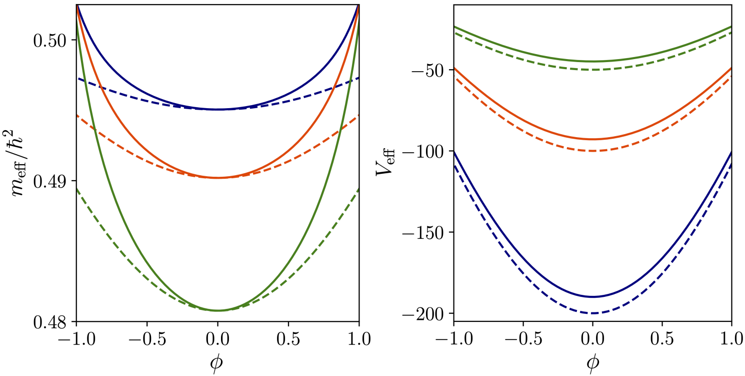
$$V_{\mathrm{eff}}(\phi) = V(\phi) + \frac{\hbar \Omega(\phi)}{2} \qquad (26)$$

with

$$\Omega(\phi)^2 = \frac{V''(\phi) - \frac{m'(\phi)}{2m(\phi)} V'(\phi)}{m(\phi)} . \qquad (27)$$

[8] C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

Quantum corrections to Josephson dynamics (II)



Effective mass (left panel) and effective potential as functions of ϕ for $U = J = 1.0$ and $N = 50$ (green lines), 100 (orange lines), and 200 (blue lines). The dashed lines represent the corresponding mean-field result. Adapted from C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

Quantum corrections to Josephson dynamics (III)

Quantum corrections do not change the position of the minimum of the effective potential $V_{\rm eff}(\phi)$, which is still located at $\phi = 0$, where also $m'_{\rm eff}(0) = 0$. In particular, small oscillations around $\phi = 0$ are harmonic, with the frequency

$$\Omega_J = \sqrt{\frac{V''_{\rm eff}(0)}{m_{\rm eff}(0)}} = \omega_J \sqrt{1 - \frac{1}{2N} \frac{UN + 6J}{\sqrt{2J(UN + 2J)}}}, \qquad (28)$$

where

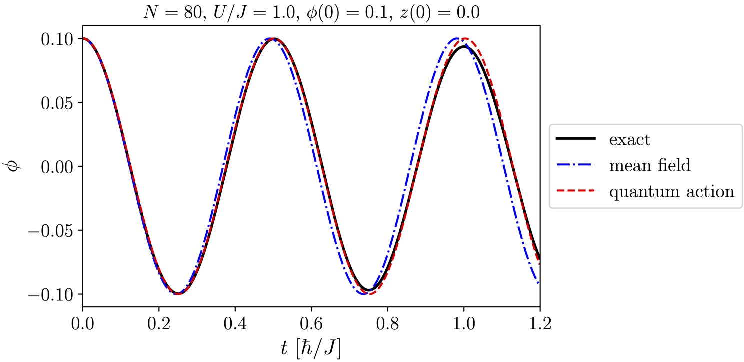
$$\omega_J = \frac{\sqrt{2J(UN + 2J)}}{\hbar} \qquad (29)$$

is the mean-field Josephson frequency.

- Exact numerical results[9] confirm the robustness of Eq. (28).
- The relative correction induced by quantum fuctuations can be of 3% for condensates with $N = 100$ atoms in realistic trapping configurations.

[9] C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

Quantum corrections to Josephson dynamics (IV)



Comparison between the exact dynamics (solid black line), the mean-field dynamics (dashed-dotted blue line), and the quantum-corrected dynamics (dashed red line) of the relative phase, for $N = 80$, $U = J = 1.0$, $\phi(0) = 0.1$, and $\dot{\phi}(0) = 0$. Adapted from C. Vianello, S. Salvatore, L.S., Int. J. Theor. Phys. **64**, 315 (2025).

Conclusions

- Quantum effective action: useful method for fields and dynamical variables.
- Provides a bridge between classical (or mean-field) dynamics and quantum fluctuations.
- Can include thermal effects perturbatively.
- Useful for theorists and experimentalists in quantum technologies.
- Work in progress: quantum effective action for optomechanics (with F. Lorenzi and M. Pelizzo).
- Work in progress: quantum effective action for resistively and capacitively shunted superconducting Josephson junction (with A. Bardin, K. Furutani, and J. Tempere).

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