Superfluids and superfluid hydrodynamics

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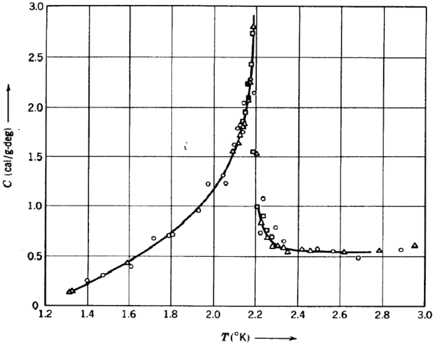
Summary

- Discovery of superfluidity
- Superfluid hydrodynamics
- Bogoliubov spectrum
- Landau criterion
- Superfluid density

Discovery of superfluidity (I)

Superfluidity is the characteristic property of a fluid with zero viscosity, which therefore flows without loss of kinetic energy.



Superfluidity was discovered in 1937 by Pyotr Kapitza, John Allen and Don Misener, who found that, at atmospheric pressure, below $T_\lambda = 2.16$ Kelvin helium 4 (^{4}He) not only remains liquid but it also shows zero viscosity. As show in the figure, at T_λ the specific heat diverges.

Discovery of superfluidity (II)

In 1938 Fritz London gave a first theoretical explanation of the superfluidity of helium 4 on the basis of Bose-Einstein condensation (BEC).
However, Lev Landau in 1941 was able to describe the superfluidity without using explicitly the BEC. Within the two-fluid model of Tisza-Landau, below T_λ Helium 4 is characterized by a inviscid superfluid component and a viscous normal component. At zero temperature only the superfluid component remains and the equations of superfluid hydrodynamics are

$$\begin{aligned} \frac{\partial}{\partial t} n + \boldsymbol{\nabla} \cdot (n \mathbf{v}) &= 0 \, , \\ m \frac{\partial}{\partial t} \mathbf{v} + \boldsymbol{\nabla} \left[\frac{1}{2} m v^2 + U(\mathbf{r}) + \mu(n) \right] &= \mathbf{0} \, , \end{aligned}$$

where $n(\mathbf{r}, t)$ is the local number density and $\mathbf{v}(\mathbf{r}, t)$ is the local velocity. $U(\mathbf{r})$ is the external potential and $\mu(n)$ is the chemical potential. These equations describe extremely well the superfluid ^{4}He, ultracold gases of alkali-metal atoms, and also several properties of superconductors.

Discovery of superfluidity (II)

In the 1950s Lars Onsager, Richard Feynman, and Alexei Abrikosov suggested that superconductors and superfluids can have quantized vortices.

In a vortex line, the number density $n(\mathbf{r})$ is zero on the line (vortex core) and around the line the velocity is quantized:

$$n(R) \simeq n(\infty)\left(1 - \frac{1}{1+\frac{R^2}{\xi^2}}\right) \quad \text{and} \quad v(R) = \frac{\hbar}{m}\frac{k}{R} ,$$

with R the distance from the vortex line and k an integer quantum number.

Quantized vortices have been experimentally observed in superfluid Helium 4, in superfluid Helium 3, in superconductors, and also in atomic BECs.[1]

[1] In 1995 Eric Cornell, Carl Wieman e Wolfgang Ketterle achieved the Bose-Einstein condensation cooling gases of alkali-metal atoms (^{87}Rb and ^{23}Na). For these bosonic systems, which are very dilute and ultracold, the critical temperature to reach the BEC is about $T_c \simeq 100$ nanoKelvin.

Superfluid hydrodynamics (I)

The zero-temperature time-dependent Gross-Pitaevskii equation

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + (N-1)g|\psi(\mathbf{r},t)|^2 \right] \psi(\mathbf{r},t) \ . \qquad (1)$$

can be rewritten as the equations of superfluid hydrodynamics. In fact, setting

$$N^{1/2}\psi(\mathbf{r},t) = n(\mathbf{r},t)^{1/2}\, e^{i\theta(\mathbf{r},t)} \ , \qquad (2)$$

and inserting this formula into Eq. (1) one finds

$$\frac{\partial}{\partial t} n + \boldsymbol{\nabla}\cdot(n\,\mathbf{v}) = 0 \ , \qquad (3)$$

$$m\frac{\partial}{\partial t}\mathbf{v} + \boldsymbol{\nabla}\left[\frac{1}{2}mv^2 + U(\mathbf{r}) + g(1-\frac{1}{N})\,n - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} \right] = \mathbf{0} \ , \qquad (4)$$

where $n(\mathbf{r},t)$ is the local number density and

$$\mathbf{v}(\mathbf{r},t) = \frac{\hbar}{m}\boldsymbol{\nabla}\theta(\mathbf{r},t) \qquad (5)$$

is the local velocity field, that is (by definition) irrotational, i.e. such that

$$\boldsymbol{\nabla}\wedge\mathbf{v} = \mathbf{0} \ . \qquad (6)$$

Superfluid hydrodynamics (II)

For a rotational and viscous fluid with quite generic zero-temperature equation of state $\mu(n, \nabla n, \nabla^2 n, ...)$ the equations of viscous hydrodynamics are given by

$$\begin{aligned} \frac{\partial}{\partial t} n + \boldsymbol{\nabla} \cdot (n \mathbf{v}) &= 0 \\ m \frac{\partial}{\partial t} \mathbf{v} + \boldsymbol{\nabla} \left[\frac{1}{2} m v^2 + U(\mathbf{r}) + \mu(n, \nabla n, \nabla^2 n, ...) \right] &= \eta \nabla^2 \mathbf{v} + m \mathbf{v} \wedge (\boldsymbol{\nabla} \wedge \mathbf{v}) \end{aligned}$$

where η is the viscosity and a rotational term appears. These equations are called zero-temperature Navier-Stokes equations. Any generic fluid (in the collisional regime) satisfies these equations.
Clearly, we recover the Gross-Pitaevskii superfluid hydrodynamics with

$$\begin{aligned} \mu(n, \nabla n, \nabla^2 n, ...) &= g(1 - \frac{1}{N})\, n - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \;, \\ \eta &= 0 \;, \\ \boldsymbol{\nabla} \wedge \mathbf{v} &= \mathbf{0} \;. \end{aligned}$$

Superfluid hydrodynamics (III)

Within the Gross-Pitaevskii superfluid hydrodynamics quantum effects are encoded not only in the equation of state but also into the properties of the local field $\mathbf{v}(\mathbf{r}, t)$:
it is proportional to the gradient of a scalar field, $\theta(\mathbf{r}, t)$, that is the angle of the phase of the single-valued complex wavefunction $\psi(\mathbf{r}, t)$.

Consequently, $\mathbf{v}(\mathbf{r}, t)$ satisfies the equation

$$\oint_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r} = \frac{\hbar}{m} \oint_{\mathcal{C}} \boldsymbol{\nabla}\theta \cdot d\mathbf{r} = \frac{\hbar}{m} \oint_{\mathcal{C}} d\theta = \frac{\hbar}{m} 2\pi\, k \tag{7}$$

for any closed contour $\mathcal{C}$, with k an integer number. In other words, the circulation is quantized in units of $\hbar/m$, and this property is strictly related to the existence of quantized vortices.

Bogoliubov spectrum (I)

We have seen that the time-dependent Gross-Pitaevskii equation can be rewritten in terms of hydrodynamic equations

$$\frac{\partial}{\partial t} n + \nabla \cdot (n \, \mathbf{v}) = 0 \, , \quad (8)$$

$$m \frac{\partial}{\partial t} \mathbf{v} + \nabla \left[\frac{1}{2} m v^2 + U(\mathbf{r}) + g(1 - \frac{1}{N}) \, n - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right] = \mathbf{0} \, . \quad (9)$$

Let us consider Eqs. (8) and (9) assuming that $U(\mathbf{r}) = 0$. We set

$$n(\mathbf{r}, t) = n_{eq} + \delta n(\mathbf{r}, t) \, , \quad (10)$$

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{0} + \delta \mathbf{v}(\mathbf{r}, t) \, , \quad (11)$$

where $\delta n(\mathbf{r}, t)$ and $\delta \mathbf{v}(\mathbf{r}, t)$ represent small variations with respect to the uniform and constant stationary configuration n_{eq}.

Bogoliubov spectrum (II)

In this way, neglecting quadratic terms in the variations (linearization) from Eqs. (8) and (9) we get the linear equations of motion

$$\frac{\partial}{\partial t}\delta n + n_{eq}\boldsymbol{\nabla}\cdot\delta\mathbf{v} = 0 \; , \tag{12}$$

$$\frac{\partial}{\partial t}\delta\mathbf{v} + \frac{c_s^2}{n_{eq}}\boldsymbol{\nabla}\delta n - \frac{\hbar^2}{4m^2 n_{eq}}\boldsymbol{\nabla}(\nabla^2\delta n) = \mathbf{0} \; , \tag{13}$$

where c_s is the zero-temperature sound velocity of the bosonic superfluid, given by

$$mc_s^2 = g(1-\frac{1}{N})\, n_{eq} \; . \tag{14}$$

The linear equations of motion can be arranged in the form of the following wave equation

$$\left[\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2 + \frac{\hbar^2}{4m^2}\nabla^4\right]\delta n(\mathbf{r},t) = 0 \; . \tag{15}$$

Bogoliubov spectrum (III)

The wave equation (15) admits monochromatic plane-wave solutions, where the frequency ω and the wave vector $\mathbf{q}$ are related by the dispersion formula $\omega = \omega(q)$ given by

$$\hbar\,\omega(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\frac{\hbar^2 q^2}{2m} + 2mc_s^2\right)} \; . \qquad (16)$$

This is the so-called Bogoliubov spectrum of elementary excitations. Notice that the Bogoliubov spectrum becomes the phonon spectrum

$$\omega(q) = c_s\, q \qquad (17)$$

in the regime $q \ll 2mc_s/\hbar$ of a small wavenumber q.

Landau criterion (I)

According to Lev Landau a superfluid is characterized by quasi-particles with a dispersion relation $E(p)$ that is not the one of free particles, i.e. $E(p) = p^2/(2m)$. Indeed, for weakly-interacting bosons we have found the dispersion relation

$$E(p) = \sqrt{\frac{p^2}{2m}\left(\frac{p^2}{2m} + 2mc_s^2\right)} \, , \tag{18}$$

which becomes the phonon spectrum $E(q) = c_s p$ for a small linear momentum $p = \hbar q$, with c_s the Bogoliubov speed of sound.
Let us consider a macroscopic particle of mass M that is moving with velocity $\mathbf{v}$ inside the superfluid. The Landau criterion says that the macroscopic particle is braked by the superfluid only if the modulus v of the velocity of the macroscopic particle is larger than the critical velocity

$$v_c = min_{\mathbf{p}} \left(\frac{E(p)}{p} \right) \, . \tag{19}$$

Clearly, in the case of the Bogoliubov spectrum one finds $v_c = c_s$, while for the free-particle spectrum one gets $v_c = 0$.

Landau criterion (II)

The Landau criterion is a consequence of the laws of conservation of linear momentum and energy in the scattering between the macroscopic particle of mass M and one quasi-particle of the superfluid:

$$\frac{1}{2}Mv^2 = \frac{1}{2}M(v')^2 + E(p) \, , \quad (20)$$

$$M\mathbf{v} = M\mathbf{v}' + \mathbf{p} \, , \quad (21)$$

where before the collision the energy and momentum of the quasi-particle are zero. Combining the two equations one finds

$$\mathbf{v} \cdot \mathbf{p} = E(p) + \frac{p^2}{2M} \, . \quad (22)$$

If the mass M is extremely large one gets

$$v = \frac{E(p)}{p \, \cos(\alpha)} \, , \quad (23)$$

where α is the angle between $\mathbf{p}$ and $\mathbf{v}$. The equality (23) holds only if v is larger than the minimum of $E(p)/p$ for any choice of p.

Superfluid density (I)

The quasi-particles with energy spectrum $E(p)$ play a crucial role in the two-fluid model of Lev Landau (1941), inspired by similar models of Lazlo Tisza (1938) for superfluids, Fritz London and Heinz London (1935) for superconductors.

According to the two-fluid model, at thermal equilibrium the superfluid system is characterized by the total number density

$$n = n_s + n_n \ , \tag{24}$$

where n_s is the superfluid number density and n_n is the normal number density.

At temperature T, in the rest frame of the superfluid, the normal current density of mass is given by

$$\mathbf{j}_n = m n_n \mathbf{v} = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \, \mathbf{p} \, f_B(E(p) - \mathbf{p}\cdot\mathbf{v}) \ , \tag{25}$$

where $\mathbf{v}$ is the velocity of the normal fluid and $f_B(E) = \frac{1}{e^{E/(k_B T)}-1}$ is the Bose distribution.

Superfluid density (II)

Assuming a small velocity $\mathbf{v}$ and Taylor-expanding the previous formula with respect to $\mathbf{v}$ to the first order one finds

$$n_n(T) = -\frac{1}{3} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{p^2}{m} \frac{d}{dE} \left(\frac{1}{e^{\frac{E(p)}{k_B T}} - 1} \right) . \qquad (26)$$

Thus, the thermal activation of quasi-particles increases the normal component of the superfluid. Clearly, the superfluid density then reads

$$n_s(T) = n - n_n(T) = n - \frac{1}{3k_B T} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{p^2}{m} \frac{e^{\frac{E(p)}{k_B T}}}{\left(e^{\frac{E(p)}{k_B T}} - 1\right)^2} \qquad (27)$$

and the critical temperature T_c of the superfluid-normal phase transition is obtained setting $n_s(T_c) = 0$.
Usually, in three spatial dimensions one finds $T_c = T_{BEC}$. However, in two spatial dimensions, for interacting bosons $T_c \neq T_{BEC} = 0$. Moreover, in one spatial dimension, for interacting bosons $T_c \neq 0$ while T_{BEC} does not exist.