Ginzburg-Landau Theory of Superconductivity

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Summary

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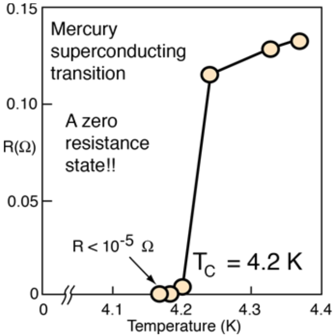
Basic superconductivity (I)

Superconductivity is a phenomenon of exactly zero electrical resistance and expulsion of magnetic flux fields occurring in certain materials when cooled below a characteristic critical temperature T_c.
It was discovered in 1911 by **Heike Kamerlingh Onnes**.



In 1957 **John Bardeen**, **Leon Cooper** and **Robert Schrieffer** suggested that in superconductivity, due to the ionic lattice, pairs of electrons behave like bosons, as somehow anticipated in 1950 by **Lev Landau** and **Vitaly Ginzburg**.

Basic superconductivity (II)

Critical temperature T_c of some superconductors at atmospheric pressure.

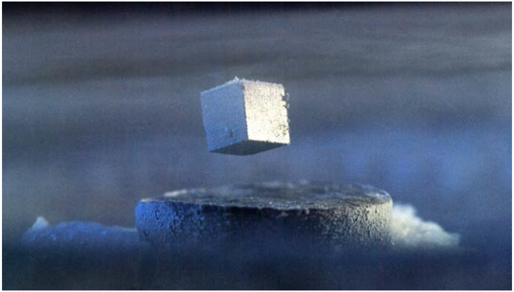
Materiale	**Symbol**	T_c **(Kelvin)**
Aluminium	${}^{27}_{13}\mathrm{Al}$	1.19
Tin	${}^{120}_{50}\mathrm{Sn}$	3.72
Mercury	${}^{202}_{80}\mathrm{Hg}$	4.16
Lead	${}^{208}_{82}\mathrm{Pb}$	7.20
Neodymium	${}^{142}_{60}\mathrm{Nb}$	9.30

In 1986 **Karl Alex Müller** and **Johannes Georg Bednorz** discovered high-T_c superconductors. These ceramic materials (cuprates) can reach the critical temperature of 133 Kelvin.

For these high-T_c superconductors the mechanisms which give rise to **pairing of electrons** are not fully understood.

Basic superconductivity (III)

Superconductors have interesting properties. For instance the levitation of a magnetic material over a supercondutor (Meissner effect).



Some technological applications of superconductors:
– MAGLEV trains, based on magnetic levitation (mag-lev);
– SQUIDS, devices which measure extremely week mgnetic fields;
– very high magnetic fields for Magnetic Resonance in hospitals.

Ginzburg-Landau phenomenological theory (I)

In 1950, seven years before the Bardeen-Cooper-Schrieffer (BCS) theory[1], Lev Landau and Vitaly Ginzburg proposed[2] a phenomenological approach to describe the superconducting phase transition. The main idea is that, close to the critical temperature, the Helmholtz free energy of a superconducting material can be written as

$$F = F_n + F_s \ , \tag{1}$$

where F_n is the contribution due to the normal component and F_s is the contibution due to the emergence of a superconducting complex order parameter ψ below a critical temperature. Ginzburg and Landau used the words ψ-theory to indicate their phenomenological theory.

[1] J. Bardeen, L.N. Cooper, and J.R. Schrieffer, Phys. Rev. **106**, 162(1957).
[2] V.L. Ginzburg and L.D. Landau, Zh. Eksp. Teor. Fiz. 20, 1064 (1950).

Ginzburg-Landau phenomenological theory (II)

Within the Ginzburg-Landau approach, for a D-dimensional system of volume L^D, the super component F_s is given by

$$F_s = \int_{L^D} d^D\mathbf{r} \left\{ a(T)\, |\psi(\mathbf{r})|^2 + \frac{b}{2}\, |\psi(\mathbf{r})|^4 + \gamma\, |\nabla\psi(\mathbf{r})|^2 \right\} , \qquad (2)$$

where

$$a(T) = \alpha\, k_B\, (T - T_c) \qquad (3)$$

is a parameter which depends on the temperature T (k_B is the Boltzmann constant) and becomes zero at the mean-field critical temperature T_c, while $b > 0$ and $\gamma > 0$ are temperature-independent phenomenological parameters.

Ginzburg-Landau phenomenological theory (III)

Assuming a real and uniform order parameter, i.e.

$$\psi(\mathbf{r}) = \psi_0 \ , \tag{4}$$

the energy functional (2) with Eqs. (3) and (4) becomes

$$\frac{F_{s0}}{L^D} = a(T)\,\psi_0^2 + \frac{b}{2}\,\psi_0^4 \ . \tag{5}$$

Minimizing F_{s0} with respect to ψ_0 one immediately finds

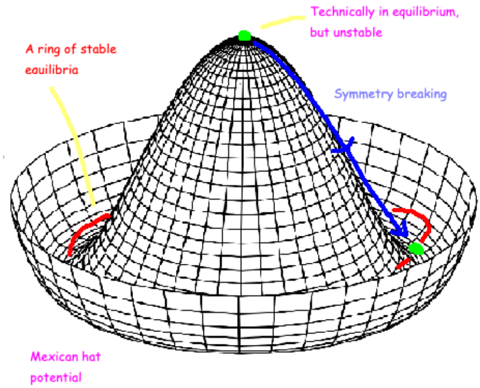
$$a(T)\,\psi_0 + b\,\psi_0^3 = 0 \ , \tag{6}$$

and consequently

$$\psi_0 = \begin{cases} 0 & \text{for } T \geq T_c \\ \sqrt{-\frac{a(T)}{b}} = \sqrt{\frac{\alpha k_B (T_c - T)}{b}} & \text{for } T < T_c \end{cases} \ . \tag{7}$$

Thus, the uniform order parameter ψ_0 becomes different from zero only below the mean-field critical temperature T_c.

Ginzburg-Landau phenomenological theory (IV)



Potential energy the Ginzburg-Landau theory with $T < T_c$, which is the typical energy landscape of second-order phase transitions: from Landau to Higgs.

Ginzburg-Landau vs Bardeen-Cooper-Schrieffer

In 1959 Lev Gor'kov showed that the phenomenological Ginzburg-Landau theory can be deduced from the microscopic theory of Bardeen-Cooper-Schrieffer (BCS, 1957). In particular, the order parameter ψ_0 can be identified with the BCS energy gap Δ_0, while the coefficients α, b and γ of the Ginzburg-Landau energy functional (2) are directly related to the parameters of the BCS Hamiltonian.

We notice, however, that the Ginzburg-Landau theory is somehow better than the the mean-field BCS theory because, in general, the Ginzburg-Landau order parameter $\psi(\mathbf{r})$ is not uniform, while the mean-field BEC energy gap Δ_0 is assumed to be uniform.

Ginzburg-Landau equation

Let us now consider the effects of a space-dependent Ginzburg-Landau order parameter $\psi(\mathbf{r})$. Extremizing the functional (2) with respect to $\psi^*(\mathbf{r})$ one gets the Euler-Lagrange equation

$$-\gamma \nabla^2\psi + a(T)\,\psi + b\,|\psi|^2\psi = 0 \; . \tag{8}$$

This equation is called Ginzburg-Landau equation, and it is formally equivalent to the zero-temperature Gross-Pitaevskii equation with the following identifications:

$$\gamma \;\to\; \frac{\hbar^2}{2m^*} \; , \tag{9}$$

$$a(T) \;\to\; -\mu \; , \tag{10}$$

$$b \;\to\; g \; . \tag{11}$$

Here $m^* > 0$ is the effective mass of the particles described by the order parameter, and $\hbar$ is the reduced Planck constant.
The connection with the BCS theory gives

$$m^* = 2m_e \; , \tag{12}$$

where m_e is the mass of the electron.

Coupling with the magnetic field (I)

In general, to describe superconductors we must take into account also the coupling with the electromagnetic field. The minimal coupling reads

$$-i\hbar\boldsymbol{\nabla} \to -i\hbar\boldsymbol{\nabla} - q\mathbf{A}(\mathbf{r}) \qquad (13)$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential, q is the charge of each of the composite bosonic-like particles described by the field $\psi(\mathbf{r})$, and $i = \sqrt{-1}$ is the imaginary unit. Thus, the Ginzburg-Landau functional becomes

$$\begin{aligned} F_s &= \int_{L^D} d^D\mathbf{r} \Big\{ a(T)\,|\psi(\mathbf{r})|^2 + \frac{b}{2}\,|\psi(\mathbf{r})|^4 \\ &+ \frac{\hbar^2}{2m^*}\,\Big|\Big(\boldsymbol{\nabla} - i\frac{q}{\hbar}\mathbf{A}(\mathbf{r})\Big)\,\psi(\mathbf{r})\Big|^2 + \frac{1}{2\mu_0}|\mathbf{B}(\mathbf{r})|^2 \Big\} \,, \end{aligned} \qquad (14)$$

Coupling with the magnetic field (II)

where

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla} \wedge \mathbf{A}(\mathbf{r}) \tag{15}$$

is the space-dependent magnetic field.
The last term in Eq. (14) takes into account the free magnetic energy of the system with μ_0 the magnetic permeability, assumed to be the vacuum one.
The minimization of the functional $F_s = F_s[\psi(\mathbf{r}), \mathbf{A}(\mathbf{r})]$ with respect to $\psi^*(\mathbf{r})$ gives the so-called Ginzburg-Landau equation

$$\left[-\frac{\hbar^2}{2m^*} \left(\boldsymbol{\nabla} - i\frac{q}{\hbar}\mathbf{A}(\mathbf{r}) \right)^2 + a(T) + b\, |\psi(\mathbf{r})|^2 \right] \psi(\mathbf{r}) = 0 \ , \tag{16}$$

that is a nonlinear Schrödinger equation with cubic nonlinearity for the order parameter $\psi(\mathbf{r})$, which contains the minimal coupling with the vector potential $\mathbf{A}(\mathbf{r})$.

Coupling with the magnetic field (III)

The supercurrent charge density $\mathbf{j}_s(\mathbf{r})$ can be obtained by considering the minimization of the functional $F_s = F_s[\psi(\mathbf{r}), \mathbf{A}(\mathbf{r})]$ with respect to $\mathbf{A}(\mathbf{r})$. In this way one finds

$$\frac{1}{\mu_0}\boldsymbol{\nabla} \wedge \mathbf{B}(\mathbf{r}) = i\frac{q\hbar^2}{2m^*}\left(\psi^*(\mathbf{r})\boldsymbol{\nabla}\psi(\mathbf{r}) - \psi(\mathbf{r})\boldsymbol{\nabla}\psi^*(\mathbf{r})\right) - \frac{q^2}{m^*}|\psi(\mathbf{r})|^2\mathbf{A}(\mathbf{r}) \,. \tag{17}$$

Remarkably, Eq. (17) can be rewritten as the familiar Ampere equation

$$\boldsymbol{\nabla} \wedge \mathbf{B}(\mathbf{r}) = \mu_0 \, \mathbf{j}_s(\mathbf{r}) \tag{18}$$

setting

$$\mathbf{j}_s(\mathbf{r}) = i\frac{q\hbar}{2m^*}\left(\psi^*(\mathbf{r})\boldsymbol{\nabla}\psi(\mathbf{r}) - \psi(\mathbf{r})\boldsymbol{\nabla}\psi^*(\mathbf{r})\right) - \frac{q^2}{m^*}|\psi(\mathbf{r})|^2\mathbf{A}(\mathbf{r}) \,, \tag{19}$$

which is identified as the supercurrent charge density.

Coupling with the magnetic field (IV)

Taking into account that the order parameter $\psi(\mathbf{r})$ describes bosons, with effective mass m^* and effective charge q, which are made of Cooper pairs (two electrons with opposite spin), we set

$$q = -2e \tag{20}$$

$$m^* = 2m_e \tag{21}$$

where $-e$ is the negative electric charge of the electron (with $e >$ the electric charge of the proton) and m_e the mass of the electron. We also introduce, in full generality, the superfluid local number density of electrons as

$$n_s(\mathbf{r}) = 2|\psi(\mathbf{r})|^2 \ . \tag{22}$$

London penetration depth (I)

Assuming a real and uniform order parameter, see Eq. (4), the supercurrent is strongly simplified and reads

$$\mathbf{j}_s(\mathbf{r}) = -\frac{q^2}{m^*}\psi_0^2 \mathbf{A}(\mathbf{r}) \,. \tag{23}$$

Clearly, from Eq. (22), in the case of a uniform and real order parameter we have

$$n_s = 2\psi_0^2 = \begin{cases} 0 & \text{for } T \geq T_c \\ \frac{2\alpha k_B (T_c - T)}{b} & \text{for } T < T_c \end{cases} \,. \tag{24}$$

In addition, using Eqs. (20), (21), and (23), the supercurrent can be rewritten as

$$\mathbf{j}_s(\mathbf{r}) = -\frac{e^2 n_s}{m_e}\mathbf{A}(\mathbf{r}) \,. \tag{25}$$

This is the London equation, obtained for the first time by the London brothers in 1935.

London penetration depth (II)

The curl of the Ampere equation (18) gives

$$-\nabla^2 \mathbf{B}(\mathbf{r}) = \mu_0 \, \boldsymbol{\nabla} \wedge \mathbf{j}_s(\mathbf{r}) \ , \tag{26}$$

taking into account that

$$\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{B}) = -\nabla^2 \mathbf{B} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{B}) = -\nabla^2 \mathbf{B} \tag{27}$$

due to the magnetic Gauss law

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \ . \tag{28}$$

Inserting Eq. (25) into Eq. (26), and using Eq. (15), we get

$$\nabla^2 \mathbf{B}(\mathbf{r}) = \frac{e^2 n_s \mu_0}{m_e} \mathbf{B}(\mathbf{r}) \ . \tag{29}$$

London penetration depth (III)

Assuming that $\mathbf{B} = \mathbf{B}(x)$ the previous equation can be written as

$$\frac{\partial^2}{\partial x^2}\mathbf{B}(x) = \frac{e^2 n_s \mu_0}{m_e}\mathbf{B}(x) \tag{30}$$

which has the following physically relevant solution for $x \geq 0$:

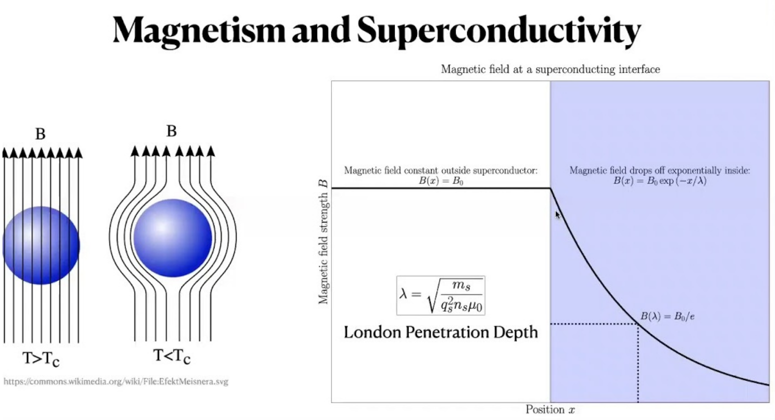
$$\mathbf{B}(x) = \mathbf{B}(0)\ e^{-x/\lambda_L}\ , \tag{31}$$

where

$$\lambda_L = \sqrt{\frac{m_e}{e^2 n_s \mu_0}} \tag{32}$$

is the so-called London penetration depth. The meaning of Eq. (31) is that inside a superconductor the magnetic field decays exponentially. This is the Meissner-Ochsenfeld effect: the expulsion of a magnetic field from a superconductor, experimentally observed for the first time in 1933.

London penetration depth (IV)



Coherence length and quantized vortices (I)

A very important characteristic length of the Ginzburg-Landau equation is the so-called coherence length

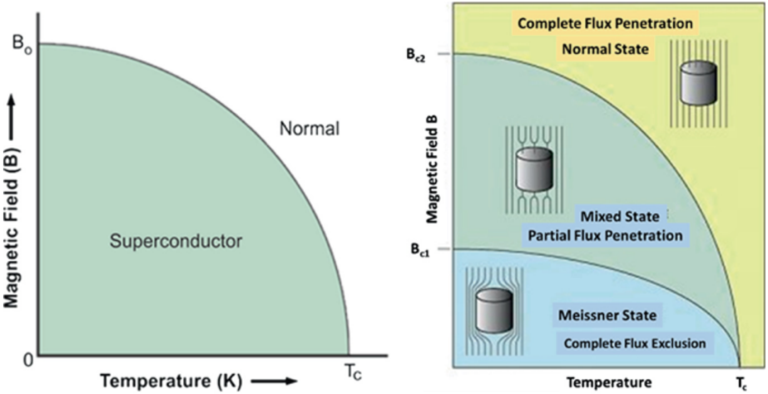
$$\xi = \sqrt{\frac{\hbar^2}{2m^*|a(T)|}} \; . \tag{33}$$

This is the distance at which there is a compensation between the gradient term and the linear term of the Ginzburg-Landau equation:

$$|-\frac{\hbar^2}{2m^*}\nabla^2\psi| \simeq \frac{\hbar^2}{2m^*\xi^2}|\psi| = |a(T)||\psi| \; . \tag{34}$$

Alexei Abrikosov in 1957 showed that the coherence length ξ is nothing else than the healing length of the quantized vortices of the Ginzburg-Landau equation.
Invoking the presence of quantized vortices Abrikosov explaned type-II superconductors, which were discovered by Rjabinin and Shubnikov in 1935.

Coherence length and quantized vortices (II)



(a) Type I (b) Type II

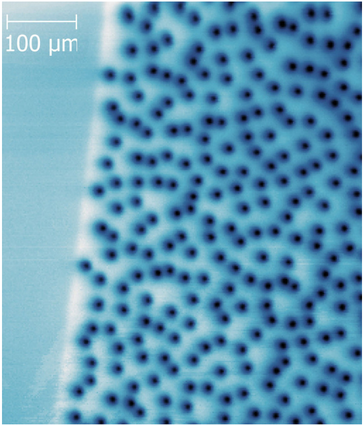
Coherence length and quantized vortices (III)

Following an earlier intuition of Ginzburg and Landau, by solving the Ginzburg–Landau equation Abrikosov found that a type-II superconductor appears when

$$\frac{\lambda_L}{\xi} > \frac{1}{\sqrt{2}} \simeq 0.7071 \; , \tag{35}$$

where λ_L is the London penetration depth and ξ is the coherence length. Under this condition, there is the formation of Abrikosov quantized vortices and the magnetic field can penetrate deep inside the superconductor.
Abrikosov found that the vortices arrange themselves into a regular array known as a vortex lattice. A similar analysis was done for the problem of vortex state in a rotating superfluid by Lars Onsager and Richard Feynman.

Coherence length and quantized vortices (IV)



Vortices in a YBCO film imaged by scanning the intensity of the magnetic field with SQUID microscopy [F.S. Wells *et al*, Sci Rep. **5**, 8677 (2015)].

Coherence length and quantized vortices (V)

Taking into account that for $T < T_c$ we have

$$\lambda_L = \sqrt{\frac{m_e}{e^2 \mu_0 n_s}} = \sqrt{\frac{m_e b}{2e^2 \mu_0 \alpha k_B (T_c - T)}} \tag{36}$$

and

$$\xi = \sqrt{\frac{\hbar^2}{4m_e |a(t)|}} = \sqrt{\frac{\hbar^2}{4m_e \alpha k_B (T_c - T)}} \, , \tag{37}$$

it follows that

$$\kappa = \frac{\lambda_L}{\xi} = \frac{m_e}{e} \sqrt{\frac{2b}{\mu_0}} \, . \tag{38}$$

Coherence length and quantized vortices (VI)

	T_c (K)	$\lambda(0)$ (nm)	$\xi(0)$ (nm)	κ
Al	1.18	1550	45	0.03
Sn	3.72	180	42	0.23
Pb	7.20	87	39	0.48
Nb	9.25	39	52	1.3
Nb_3Ge	23.2	3	90	30
YNi_2B_2C	15	8.1	103	12.7
K_3C_{60}	19.4	2.8	240	95
$YBa_2Cu_3O_{7-\delta}$	91	1.65	156	95