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Physics of solitons and solitons in optical fibers

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History of solitons (I)

The **19th century** and the **firts half of the 20th century** were the triumph of **linear physics**: theoretical approaches were trying to avoid nonlinearities treating them as perturbations of linear theories.

In the **second half of the 20th century** the importance of an intrinsic analysis of **nonlinear phenomena** has been gradually understood and led to two concepts that revolutionalised prevoius ideas:

- *Strange attractor*: linked to chaos of systems with a small number of degrees of freedom and that are described by deterministic equations
- *Soliton*: linked to collective effects leading to spatially coherent structures resulting in self-organization of systems with a very large number of degrees of freedom

So, what is a *soliton*?

- It's a **solitary wave** (=spatially localized) with spectacular **stability** properties
- Its name make it seem a particle: it's a **local maximum of energy density preserving shape and velocity** as it moves
- It corresponds to a solution of a classical field equation which simultaneously exhibits **wave** and **quasi-particle** properties
- Equations with soliton solutions are **completely integrable systems** with an **infinite** number of degrees of freedom [1]

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

History of solitons (II)

- 1834: **first observation** [2] by **John Scott Russell**, an hydrodinamic engineer, who devoted 10 years of research to solitons (only to find that linearised approaches were showing their non-existence)
- 1895: **theory** describing solitons thanks to an equation derived by **Korteweg** and **de Vries** [3]
- 1953: **numerical experiment** performed by **Fermi, Pasta** and **Ulam** with one of the first computer in Los Alamos (the result seemed to contradict thermodynamics)
- 1965: **solitons** explain the latter phenomenon, as demonstrated by **Zabusky** and **Kruskal** [4]
- 1973: Non Linear Schrodinger Equation (**NLSE**) for an optical fiber was proposed by **Hasegawa** and **Tappert** [5,6]
- 1980: **experimental checks** by **Mollenauer, Stolen** and **Islam** [7]

Solitary wave recreated in the Union Canal near Edimburgh in 1995, 161 years after John Scott Russell's discovery of what he called 'The great solitary wave' (Picture: Chris Eilbeck & Heriot-Watt University, 1995)

[2] Russell, J.S., *Rep. 14th Meet. Br. Ass. Adv. Sc.* **25**, 311-90 (1844).
[3] Korteweg, D.J., de Vries, G., *Phyl. Mag. 5th Series* **36**, 422-43 (1895).
[4] Zabusky, N.J., Kruskal, M.D., *Phys. Rev. Lett.* **15**, 240-3 (1965).
[5] Hasegawa, A., Tappert, F., *Appl. Phys. Lett.* **23.3**, 142-144 (1973).
[6] Hasegawa, A.J., Tappert, F., *Appl. Phys. Lett.* **23**, 171-2 (1973).
[7] Mollenauer, L.F., Stolen, R.H., Islam, M.N., *Phys. Rev. Lett.* **45**, 1095-8 (1980).

The discovery (I)

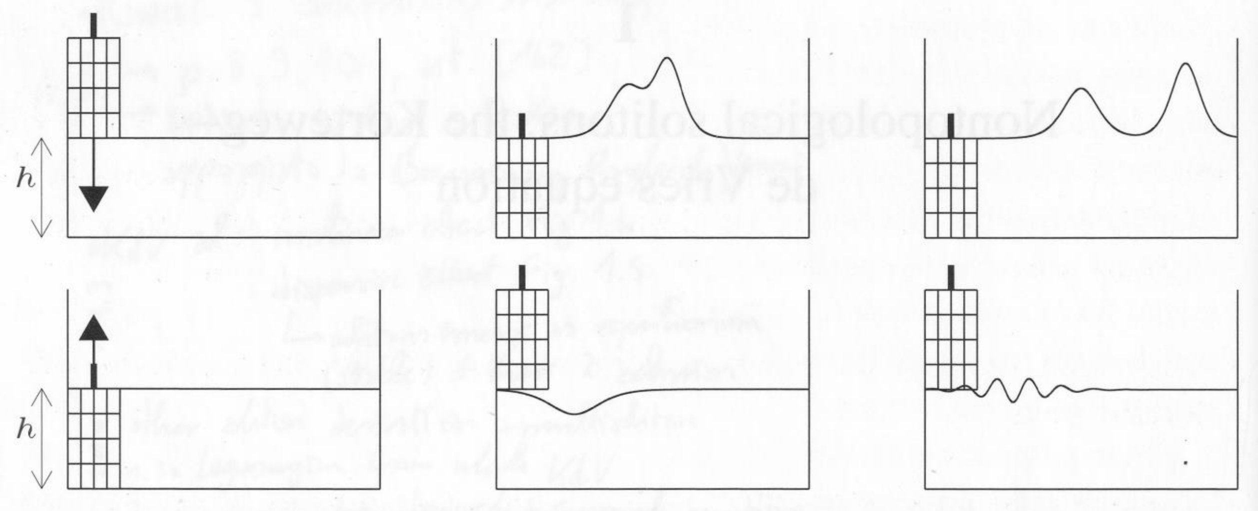
- **First observation** of a soliton was made in **1834** by the hydrodynamic engineer **John Scott Russell** while he was riding his horse along a canal near Edimburgh. The description that he gave was the following:

> *"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel, apparently without change of form or diminution of speed. I followed it on horseback and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the winding of the channel. Such in the month of August 1834 was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."*

The discovery (II)

- Russell then devoted 10 years of **research** to solitons experimenting in canals, rivers, lakes and his backyard 10 m long tank, observing the following features [2]:
  - Depending on its **amplitude**, the initial perturbation can create one, two or several solitary waves;
  - Nonlinear waves have a **speed** higher than the speed $c_0 = \sqrt{gh}$ of long-wavelength linear waves $v = c_0(1 + A\eta)$ where $A > 0$;
  - There are no solitary waves with a negative amplitude.

- **Airy** and **Stokes** strongly **criticized** Russell's work, who stopped his research in this field
- **De Boussinesq** theory of shallow water waves had solutions which **agreed** whit Russell's observations and these results were confirmed by **Lord Rayleigh** in 1876
- **De Saint Venant** in 1885 established a correct mathematical theory for these phenomena, which **explained Airy and Stokes' mistakes**



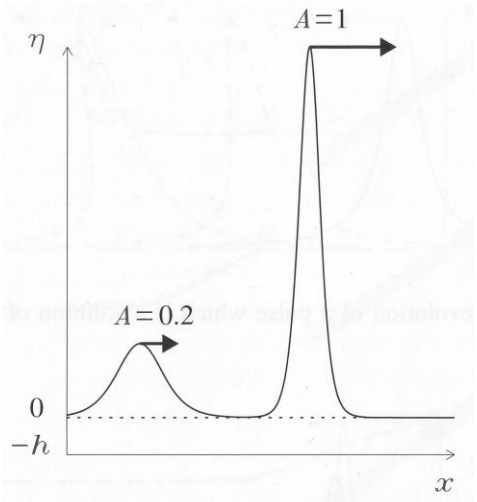
Schematic picture of the time evolution of a perturbation of the water surface in a reservoir, driven by a piston moving downward or upward. [1]

[2] Russell, J.S., *Rep. 14th Meet. Br. Ass. Adv. Sc.* **25**, 311-90 (1844).

The Korteweg-de Vries equation (I)

The Korteweg-de Vries (**KdV**) equation:

- is **derived** from the Euler equation for a nonviscous and incrompressible fluid, the boundary conditions at the bottom and at the surface and the assumption of an irrotational flow;
- is **valid** in the weakly nonlinear case;
- $c_0 = \sqrt{gh}$ speed of long-wavelength linear waves, h depth of the fluid, $\eta(x,t)$ height of the surface above its equilibrium level.



Comparison between solitons having different amplitudes A. The left pulse, having moderate amplitude and speed, is broader than the faste right pulse, being $L = \sqrt{2/A}$. [1]

$$\frac{1}{c_0}\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{3}{2h}\eta\frac{\partial \eta}{\partial x} + \frac{h^2}{6}\frac{\partial^3 \eta}{\partial x^3} = 0$$

Frame moving at c_0: $X = x - c_0 t$ and $T = t$

$$\frac{1}{c_0}\frac{\partial \eta}{\partial T} + \frac{3}{2h}\eta\frac{\partial \eta}{\partial X} + \frac{h^2}{6}\frac{\partial^3 \eta}{\partial X^3} = 0$$

Dimensionless variables: $\phi = \frac{\eta}{h}, \xi = \frac{X}{X_0}, \tau = \frac{T}{T_0}$

$$\frac{\partial \phi}{\partial \tau} + 6\phi\frac{\partial \phi}{\partial \xi} + \frac{\partial^3 \phi}{\partial \xi^3} = 0$$

Solution $A > 0$

$$\phi = A\,\mathrm{sech}^2\left[\sqrt{\frac{A}{2}}(\xi - 2A\tau)\right]$$

Solution in the laboratory frame $\eta_0 > 0$

$$\eta = \eta_0\,\mathrm{sech}^2\left[\frac{1}{2h}\sqrt{\frac{3\eta_0}{h}}\left(x - c_0\left[1 + \frac{\eta_0}{2h}\right]t\right)\right]$$

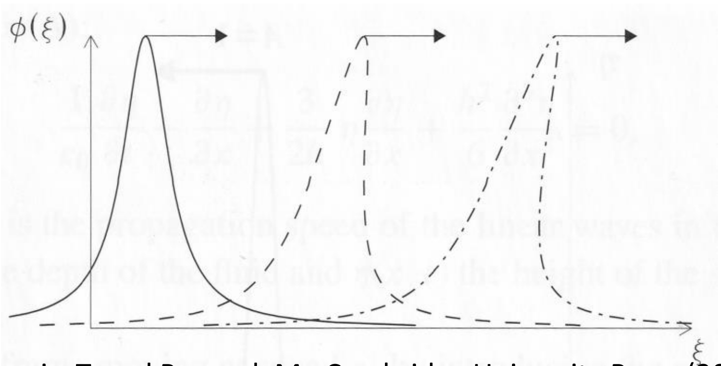
[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

The Korteweg-de Vries equation (II)

$$\frac{\partial \phi}{\partial \tau} + 6\phi \frac{\partial \phi}{\partial \xi} + \frac{\partial^3 \phi}{\partial \xi^3} = 0$$

$$\frac{\partial \phi}{\partial \tau} + \phi \frac{\partial \phi}{\partial \xi} = 0$$

- Also called **Burgers-Hopf** equation;
- In analogy with a linear differential equation, ϕ is the speed of each component of the signal;
- The parts of the signal with largest amplitude ϕ move faster than the parts with a smaller amplitude;
- The **nonlinear term** $\phi(\partial\phi/\partial\xi)$ tends to promote the formation of **steep fronts** (or shock waves).

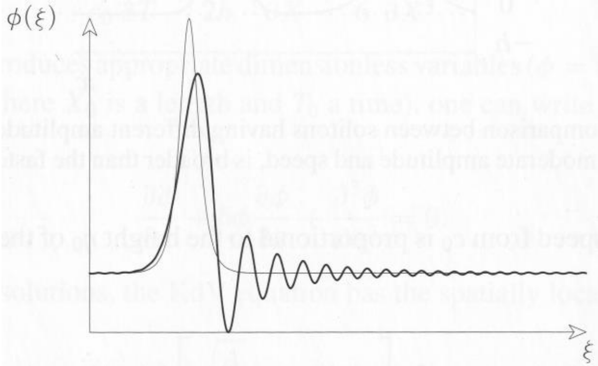


[1]

$$\frac{\partial \phi}{\partial \tau} + \frac{\partial^3 \phi}{\partial \xi^3} = 0$$

- **Linearization** of the KdV;
- Plane wave solutions of the form $\phi = Ae^{i(q\xi-\omega\tau)}$ provided the dispersion relation $\omega = -q^3$;
- The waves have a phase velocity $v_\varphi = \omega/q \propto q^2$;
- Thus we're dealing with a **dispersive medium** in which the Fourier components of a pulse propagate at **different speeds**, causing the **broadening of the pulse**.

Equilibrium between nonlinearity and dispersion gives rise to a permanent profile **soliton** solution: this equilibrium is **stable**



[1]

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

The Korteweg-de Vries equation (III)

Permanent profile solutions propagate at speed v while preserving their shape, therefore they can be written as $\phi(\xi,\tau) = \phi(\xi - v\tau) = \phi(z)$ and the KdV becomes, using the notation $\phi_z = \partial\phi/\partial z$:

$$-v\phi_z + 6\phi\phi_z + \phi_{zzz} = 0$$

Collecting a derivative in z

$$\frac{d}{dz}\left(-v\phi + 3\phi^2 + \phi_{zz}\right) = 0$$

Integrating in z

$$\phi_{zz} + 3\phi^2 - v\phi + c_1 = 0$$

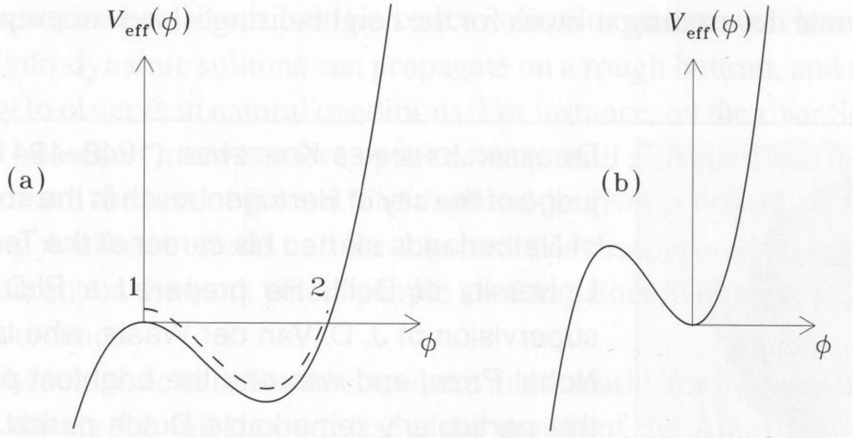
Multipling by ϕ_z and integrating in z

$$\frac{1}{2}\phi_z^2 + \phi^3 - \frac{1}{2}v\phi^2 + c_1\phi = c_2$$

Energy conservation for a particle with $m = 1$

$$\frac{1}{2}\phi_z^2 + V_{eff}(\phi) = c_2 \quad \text{with} \quad V_{eff}(\phi) = \phi^3 - \frac{1}{2}v\phi^2 + c_1\phi$$

Shape of the effective potential $V_{eff}(\phi)$ for $v > 0$ (a) and $v < 0$ (b). [1]



[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

The Korteweg-de Vries equation (IV)

Soliton solutions are **spatially localised solutions**, meaning that $\phi, \phi_z, \phi_{zz} \to 0$ when $z \to \infty$, thus it implies $c_1 = c_2 = 0$ and

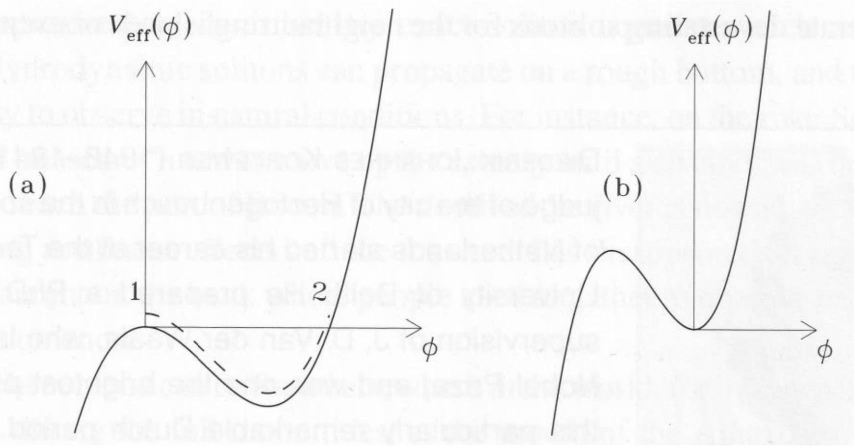
$$\frac{1}{2}\phi_z^2 + \phi^3 - \frac{1}{2}v\phi^2 = 0$$

Variables separation

$$dz = \frac{d\phi}{\sqrt{v\phi^2 - 2\phi^3}}$$

Integrating with the change of variable $\phi = \frac{v\,\mathrm{sech}^2\,u}{2}$

$$\phi = \frac{v}{2}\mathrm{sech}^2\left(\sqrt{\frac{v}{4}}z\right)$$



Shape of the effective potential $V_{eff}(\phi)$ for $v > 0$ (a) and $v < 0$ (b). [1]

Soliton solution

- $v > 0$: particle at rest can leave the origin $\phi = 0$ only moving towards the positive side (no negative ϕ amplitude solutions);
- $v < 0$: no constant energy bounded motion for a particle at rest leaving the origin $\phi = 0$

$\Longrightarrow$ **hydrodynamic solitons are always supersonic**

It can be demonstrated that dropping the spatially localised solutions requirement, **cnoidal waves** become the solution that is physically relevant for hydrodinamic waves:

$$\phi = \phi_0 - \frac{k^2q^2}{2}\mathrm{cn}^2\left(\frac{qx}{2}, k\right) \quad \text{with} \quad \mathrm{cn}(x,k) = \begin{cases} \cos x\,, & k = 0 \\ \mathrm{sech}\, x\,, & k = 1 \end{cases}$$

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

The Korteweg-de Vries equation (V)

Besides permanent profile solutions, the KdV equation also has an infinity of other solutions called **multisoliton solutions** because, when $|t| \to \infty$, they tend toward a superposition of **several well separated solitons**.

Let us consider the **two soliton solution** where we introduce $X_i = K_i\left(\xi - 4K_i^2\tau\right),\ i = 1{,}2$ whit $K_1 > K_2$:

$$\phi = \frac{2(K_1^2 - K_2^2)}{(K_1 \coth X_1 - K_2 \tanh X_2\ )^2}\left(\frac{K_1^2}{\sinh^2 X_1} + \frac{K_2^2}{\cosh^2 X_2}\right)$$

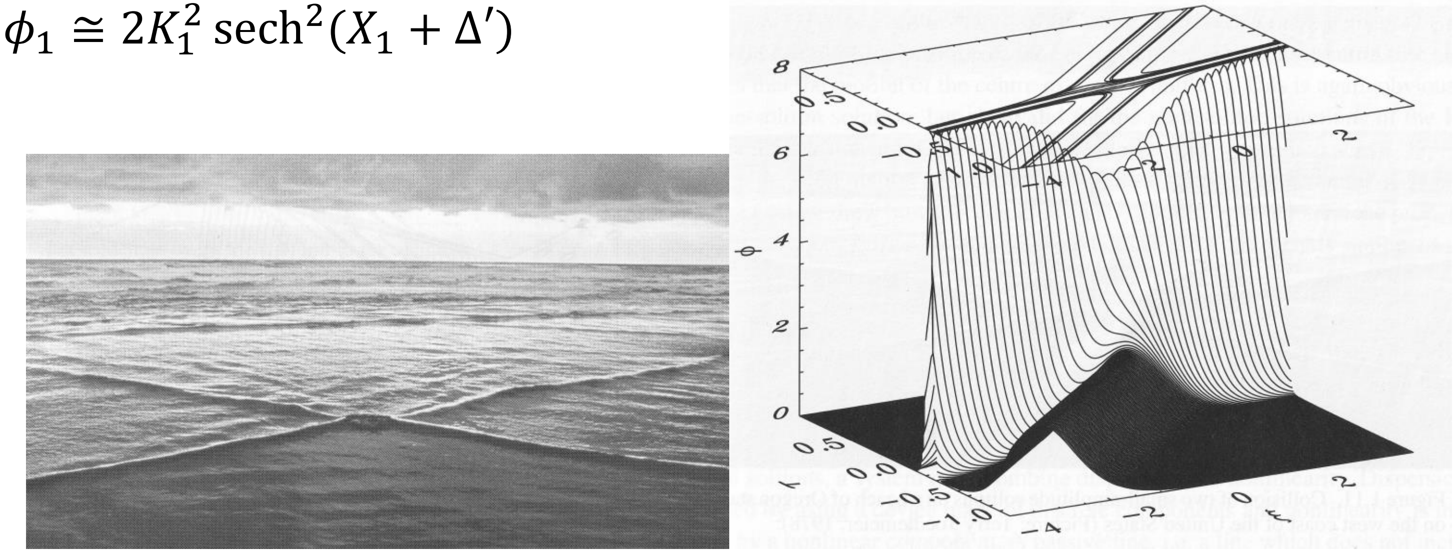
- Solution ϕ_2 in the vicinity of $\xi = 4K_2^2\tau$ when $\tau \to -\infty$, is identical to a single KdV soliton with $\mathrm{A}_2 = 2K_2^2$ and $v_2 = 4K_2^2$:

$$\phi_2 \cong 2K_2^2\,\mathrm{sech}^2(X_2 - \Delta)$$

- Solution ϕ_1 in the vicinity of $\xi = 4K_1^2\tau$ when $\tau \to -\infty$, is identical to a single KdV soliton with $\mathrm{A}_1 = 2K_1^2$ and $v_1 = 4K_1^2$:

$$\phi_1 \cong 2K_1^2\,\mathrm{sech}^2(X_1 + \Delta')$$

- In $\tau \to -\infty$ we have two different solitons and since $v_1 > v_2$ we get that in $\tau \to +\infty$ **soliton 1 has passed soliton 2**:
  - Each soliton has kept its velocity but experiences a phase shift;
  - The collision isn't a simple superposition: amplitudes don't sum up.



Solitons collision: small amplitude sea waves and different amplitudes soliton simulation. [1]

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

The chain of coupled pendula

Let us consider a **chain of coupled pendula**, moving around a common axis and linked to the neighbors by torsional springs:

$$\mathcal{H} = \sum_n \frac{I}{2}\left(\frac{d\theta_n}{dt}\right)^2 + \frac{C}{2}(\theta_n - \theta_{n-1})^2 + mgl(1 - \cos\theta_n)$$

Considering the momentum $p_n = I\dot{\theta}_n$ and the hamiltonian equations

$$\frac{d\theta_n}{dt} = \frac{\partial\mathcal{H}}{\partial p_n} \quad \text{and} \quad \frac{dp_n}{dt} = -\frac{\partial\mathcal{H}}{\partial\theta_n}$$

we get the nonlinear coupled differential equations

$$I\frac{d^2\theta_n}{dt^2} - C(\theta_{n+1} + \theta_{n-1} - 2\theta_n) + mgl\sin\theta_n = 0.$$

Solution are found in the **continuum limit approximation**: $\theta_n - \theta_{n-1} \ll 1$ allows to Taylor expand

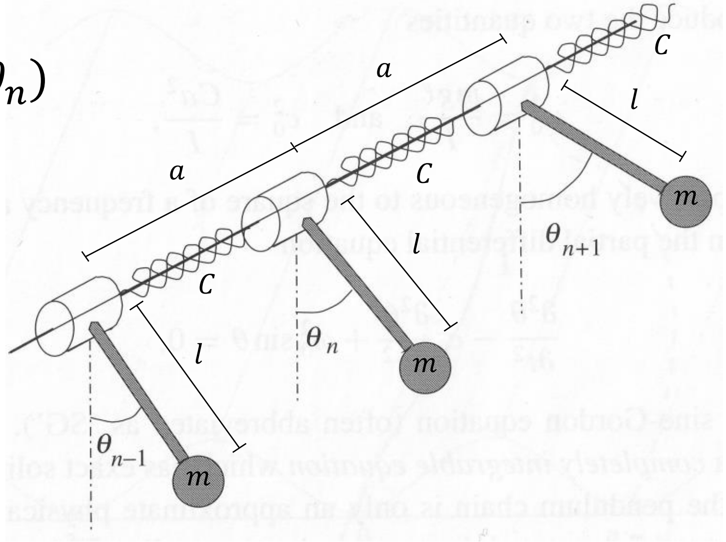
$$\theta_{n+1} + \theta_{n-1} - 2\theta_n \cong a^2\frac{\partial^2\theta}{\partial x^2} + \mathcal{O}\left(a^4\frac{\partial^4\theta}{\partial x^4}\right)$$

and introducing the quantities

$$\omega_0^2 = \frac{mgl}{I} \quad \text{and} \quad c_0^2 = \frac{Ca^2}{I}$$

we get the partial differential equation known as the **sine-Gordon (SG) equation**:

$$\frac{\partial^2\theta}{\partial t^2} - c_0^2\frac{\partial^2\theta}{\partial x^2} + \omega_0^2\sin\theta = 0$$



Scheme of a chain of coupled pendula. [1]

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

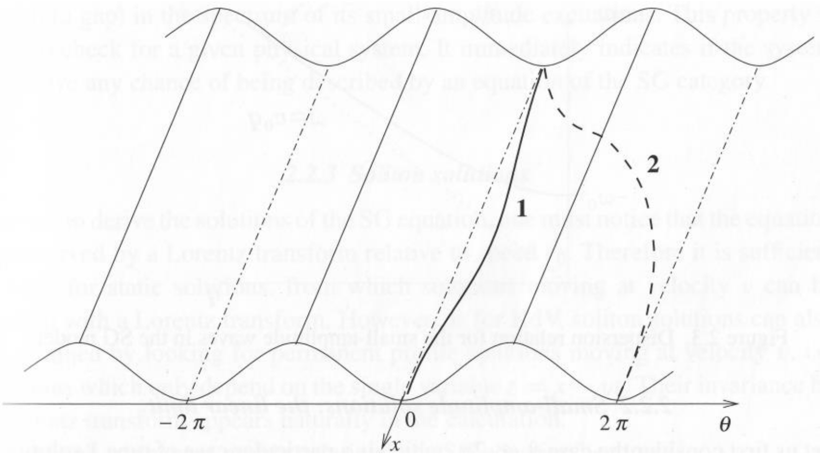
The sine-Gordon equation (I)

A pendulum of the chain is subject to the potential

$$V(\theta) = mgl(1 - \cos\theta)$$

so that the energy landscape oscillates regularly, with several energetically **degenerate ground states in $\boldsymbol{\theta = 2p\pi, p \in \mathbb{Z}}$**.
This suggests the existence of **families of solutions** of the SG equation:

- Case 1
  Solutions staying within a **single** potential valley
  $$\lim_{x\to+\infty}\theta - \lim_{x\to-\infty}\theta = 0$$
- Case 2
  Solutions **moving** from one valley to the other
  $$\lim_{x\to+\infty}\theta - \lim_{x\to-\infty}\theta = 2p\pi \quad (p \neq 0)$$

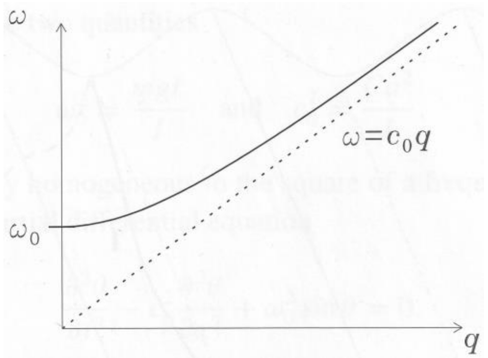


Topology of the potential energy landscape of the SG model. [1]

Solutions are **topologically different** because their difference is a property of the whole solution: one must look at the whole chain of pendula to see a full turn from one end to the other of case 2.

The particular case of $\theta \ll 2\pi$ falls into case 1 and it's called the **small amplitude** regime.
We can take the **linear limit** $\sin\theta \approx \theta$ such that the SG equation reduces to

$$\frac{\partial^2\theta}{\partial t^2} - c_0^2\frac{\partial^2\theta}{\partial x^2} + \omega_0^2\theta = 0$$



with plane waves solutions $\theta = \theta_0 e^{i(qx-\omega t)} + c.c.$ and dispersion relation $\omega^2 = \omega_0^2 + c_0^2 q^2 \propto q^2 \Longrightarrow$ **dispersive waves**

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

The sine-Gordon equation (II)

Let us look for **permanent profile solutions** of the SG equation, moving at velocity v and depending on the variable $z = x - vt$:

$$v^2 \frac{d^2\theta}{dz^2} - c_0^2 \frac{d^2\theta}{dz^2} + \omega_0^2 \sin\theta = 0$$

Collecting the derivative in z

$$\frac{d^2\theta}{dz^2} = \frac{\omega_0^2}{c_0^2 - v^2} \sin\theta$$

Multiplying by $d\theta/dz$ and integrating in z

$$\frac{1}{2}\left(\frac{d\theta}{dz}\right)^2 = -\frac{\omega_0^2}{c_0^2 - v^2}\cos\theta + C_1$$

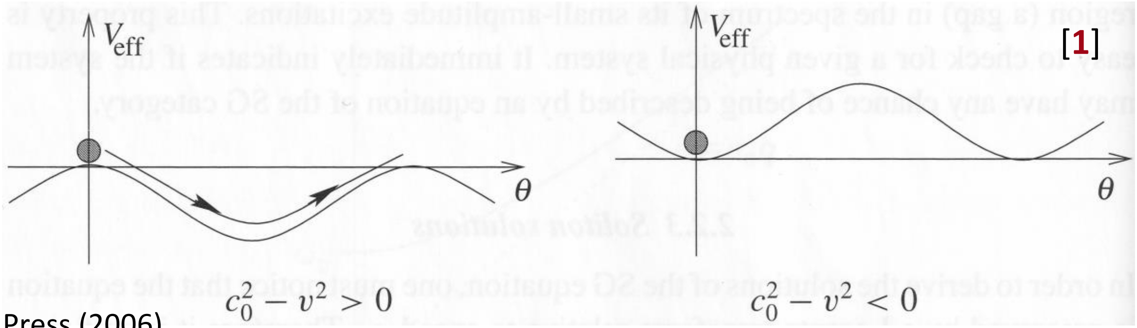
Imposing $\lim_{|z|\to\infty} \theta(z) = 0\ (mod\ 2\pi)$ and $\lim_{|z|\to\infty} d\theta/dz = 0$ we get $C_1 = \omega_0^2/(c_0^2 - v^2)$

$$\frac{1}{2}\left(\frac{d\theta}{dz}\right)^2 - \frac{\omega_0^2}{c_0^2 - v^2}(1 - \cos\theta) = 0$$

The solution $\theta(z)$ describes the motion of a fictitious particle having zero total energy in the potential

$$V_{eff}(\theta) = -\frac{\omega_0^2}{c_0^2 - v^2}(1 - \cos\theta)$$

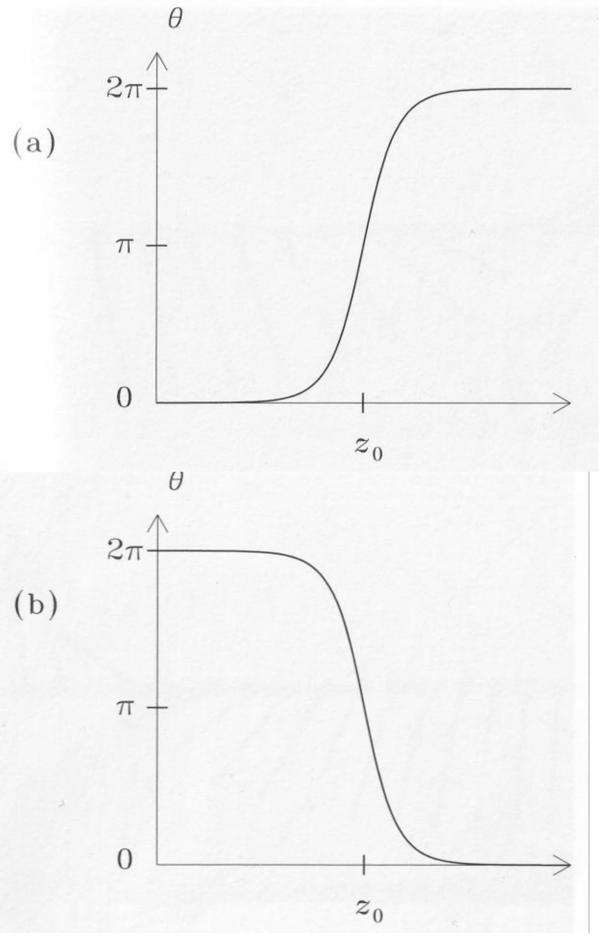
Possible motion of a particle leaving $\theta = 0$ at rest $\Longrightarrow$ **solitons travel at $v < c_0$**



A particle initially at rest in $\theta = 0$ cannot move $\Longrightarrow$ **no permanent profile solution within same valley**

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

The sine-Gordon equation (III)



Soliton (a) and antisoliton (b) solutions of the SG equation. [1]

$$\frac{1}{2}\left(\frac{d\theta}{dz}\right)^2 - \frac{\omega_0^2}{c_0^2 - v^2}(1-\cos\theta) = 0$$

Variables separation

$$\frac{\sqrt{2}\omega_0}{\sqrt{c_0^2 - v^2}}dz = \pm\frac{d\theta}{\sqrt{1-\cos\theta}}$$

Integrating in z and manipulating the right hand side

$$\frac{\sqrt{2}\omega_0}{\sqrt{c_0^2 - v^2}}(z - z_0) = \pm\int\frac{d\theta}{\sqrt{2}\sin(\theta/2)} \qquad (0 < \theta < 2\pi)$$

Using $t = \tan(\theta/4)$

$$\int\frac{d\theta}{\sqrt{2}\sin(\theta/2)} = \frac{1}{\sqrt{2}}\int\frac{4dt}{1+t^2}\frac{1+t^2}{2t} = \sqrt{2}\int\frac{dt}{t} = \sqrt{2}\ln t$$

$$\frac{\omega_0}{\sqrt{c_0^2 - v^2}}(z - z_0) = \pm\ln\tan\frac{\theta}{4}$$

Expliciting θ

$$\theta = 4\arctan\exp\left[\pm\frac{\omega_0}{c_0}\frac{z - z_0}{\sqrt{1 - v^2/c_0^2}}\right]$$

with $z = x - vt$

Validity condition: $v^2 < c_0^2$

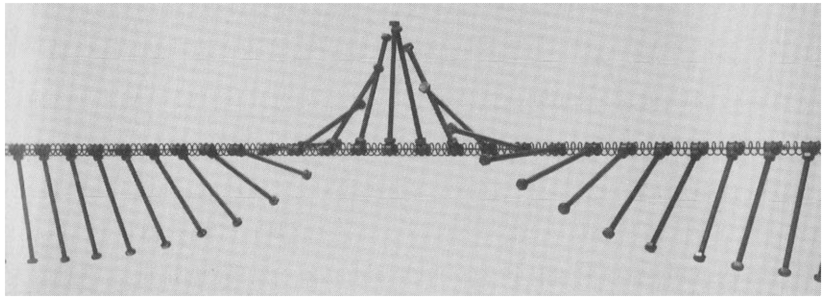
- Soliton
- + sign
- Kink

- Antisoliton
- - sign
- Antikink

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

The sine-Gordon equation (IV)

The pendulum chain exhibits a 2π torsion when it carries a soliton (Picture: Thierry Dauxois & Bruno Issenman, 2003)



A frozen *kink*, which was naturally created by snow fallen on a horizontal bar (Picture: Thierry Cretegny, 2001)

The sine-Gordon equation (V)

- Solitons and antisolitons differ by their **topological charge Q**, which is a **conserved quantity** (stability)

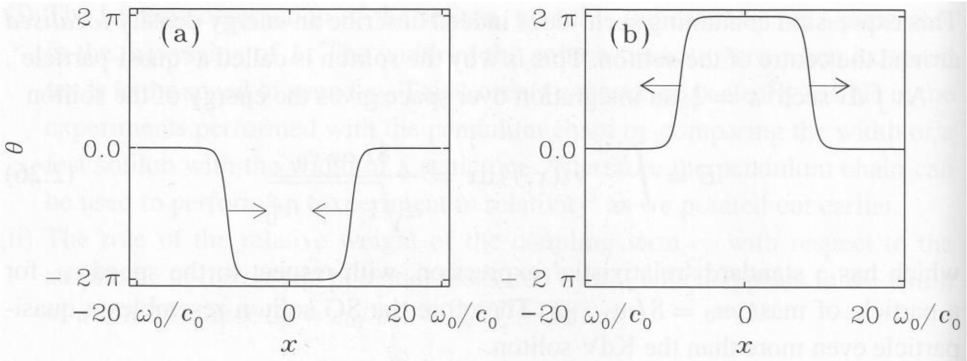
$$Q = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\partial\theta}{\partial x} dx = \frac{1}{2\pi}\left[\lim_{x\to+\infty}\theta(x,t) - \lim_{x\to-\infty}\theta(x,t)\right] = \begin{cases} +1 & \textit{soliton} \\ -1 & \textit{antisoliton} \end{cases}$$

- Validity of solutions $\theta = 4\arctan\exp\left[\pm\frac{x-vt}{L}\right]$ in a discrete chain are measured through their **spatial extent** $L = \frac{c_0}{\omega_0}\sqrt{1-\frac{v^2}{c_0^2}}$
  - Width of the soliton tends to zero when $v \to c_0$: **Lorentz contraction** as in relativity;
  - The width of a soliton at rest is $L_0 = \frac{c_0}{\omega_0} = a\sqrt{\frac{C}{mgl}}$: continuum limit is valid if $L_0/a \gg 1 \Rightarrow \mathrm{C} \gg mgl$ **strong coupling**.
- In the continuum limit, the **energy of the soliton** is obtained from the hamiltonian density $\hbar = \mathcal{H}/a \propto \mathrm{sech}^2(z-z_0)$:

$$E = \int_{-\infty}^{+\infty} \hbar(x,t)dx = \frac{8I\omega_0 c_0}{a\sqrt{1-v^2/c_0^2}}$$

- The inverse scattering method is a systematic approach to get **multisoliton solutions** (arbitrary number of solitons and antisolitons)
  - Soliton-antisoliton: attractive (if bound it's called *breather*)
  - Soliton-soliton and antisoliton-antisoliton: repulsive

Multisoliton solution: soliton and antisoliton colision. [1]



[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

Nonlinear waves in the pendulum chain (I)

Let us consider again the **SG equation** in the continuum limit, where we use the shortened notation $\frac{\partial^2\theta}{\partial t^2} = \theta_{tt}$ and $\frac{\partial^2\theta}{\partial x^2} = \theta_{xx}$

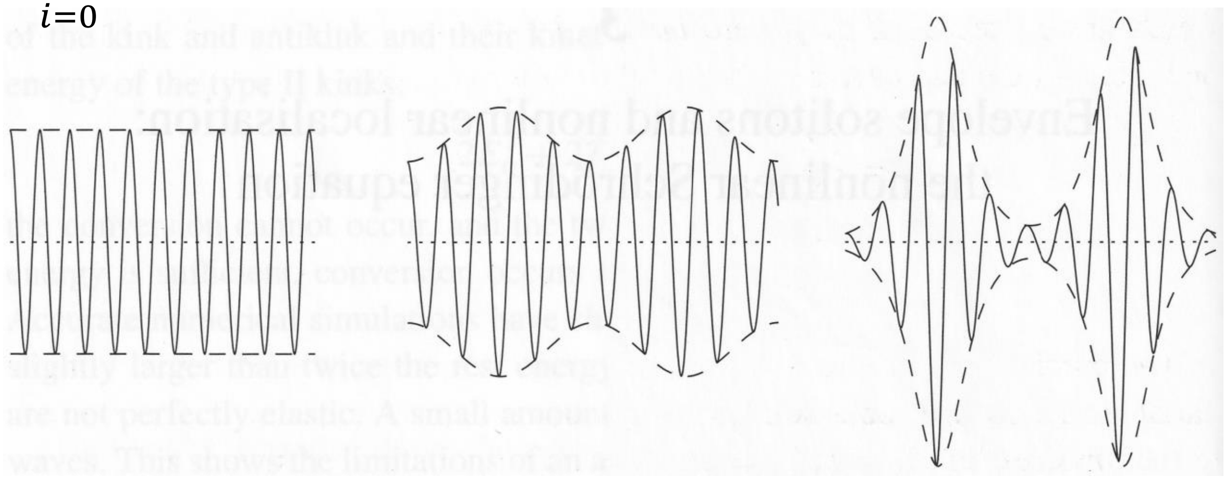
$$\theta_{tt} - c_0^2\theta_{xx} + \omega_0^2 \sin\theta = 0.$$

Let us analyse the **medium-amplitude regime**, where a weak **nonlinearity** comes into play from the Taylor expansion

$$\sin\theta = \theta - \frac{\theta^3}{6} + \mathcal{O}(\theta^5).$$

We expect plane waves to **self-modulate** due to nonlinearity: wave packets behave like solitons made of a **carrier** wave with fast time-space variation and of a slower **envelope** signal: **multiple scale expansion method** with θ as a perturbative series of functions ϕ_i of the independent variables $T_i = \varepsilon^i t$ and $X_i = \varepsilon^i x$:

$$\theta(x,t) = \varepsilon \sum_{i=0}^{\infty} \varepsilon_i\, \phi_i(X_0, X_1, X_2, \dots, T_0, T_1, T_2, \dots).$$



Successive stages of the self-modulation of a plane wave: solid line is the carrier wave, dashed line is the envelope. [1]

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

Nonlinear waves in the pendulum chain (II)

Introducing the notation $D_i = \partial/\partial T_i$ and $D_{X_i} = \partial/\partial X_i$ we get

$$\frac{\partial^2}{\partial t^2} = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots)^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \cdots,$$

$$\frac{\partial^2}{\partial x^2} = \left(D_{X_0} + \varepsilon D_{X_1} + \varepsilon^2 D_{X_2} + \cdots\right)^2 = D_{X_0}^2 + 2\varepsilon D_{X_0} D_{X_1} + \varepsilon^2\left(D_{X_1}^2 + 2D_{X_0} D_{X_2}\right) + \cdots.$$

Inserting the expansion $\theta = \varepsilon\phi_0 + \varepsilon^2\phi_1 + \varepsilon^3\phi_2 + \cdots$ into the SG equation $\theta_{tt} - c_0^2\theta_{xx} + \omega_0^2\theta - \omega_0^2\theta^3/6 = 0$ we get:

- **At order ε**:

  $$\left(D_0^2 - c_0^2 D_{X_0}^2 + \omega_0^2\right)\phi_0 = \hat{L}\phi_0 = 0$$

  linear in ϕ_0 with **plane waves** solutions

  $$\phi_0 = A(X_1, T_1, X_2, T_2, \dots)e^{i(qX_0 - \omega T_0)} + c.c.$$

  and dispersion relation

  $$\omega^2 = \omega_0^2 + c_0^2 q^2.$$

  Plane waves can be **stable** ($PQ < 0$) or subject to **modulational instability** ($PQ > 0$), where a small perturbation leads to a train of solitons, depending on the system condition.

Nonlinear waves in the pendulum chain (III)

Introducing the notation $D_i = \partial/\partial T_i$ and $D_{X_i} = \partial/\partial X_i$ we get

$$\frac{\partial^2}{\partial t^2} = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots)^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \cdots,$$

$$\frac{\partial^2}{\partial x^2} = \left(D_{X_0} + \varepsilon D_{X_1} + \varepsilon^2 D_{X_2} + \cdots\right)^2 = D_{X_0}^2 + 2\varepsilon D_{X_0} D_{X_1} + \varepsilon^2\left(D_{X_1}^2 + 2D_{X_0} D_{X_2}\right) + \cdots.$$

Inserting the expansion $\theta = \varepsilon\phi_0 + \varepsilon^2\phi_1 + \varepsilon^3\phi_2 + \cdots$ into the SG equation $\theta_{tt} - c_0^2\theta_{xx} + \omega_0^2\theta - \omega_0^2\theta^3/6 = 0$ we get:

- **At order ε^2**:

$$D_0^2\phi_1 + 2D_0 D_1\phi_0 - c_0^2 D_{X_0}^2\phi_1 - 2c_0^2 D_{X_0} D_{X_1}\phi_0 + \omega_0^2\phi_1 = 0$$

that we rewrite separating ϕ_0 and ϕ_1 and introducing $\sigma = qX_0 - \omega T_0$

$$\hat{L}\phi_1 = -2D_0 D_1\phi_0 - 2c_0^2 D_{X_0} D_{X_1}\phi_0 = 2i\omega\frac{\partial A}{\partial T_1}e^{i\sigma} + 2iqc_0^2\frac{\partial A}{\partial X_1}e^{i\sigma} + c.c.$$

We found a linear equation driven by resonance terms $e^{i\sigma}$, who make the responce grow linearly with time: we thus need the **solvability condition**:

$$\frac{\partial A}{\partial T_1} + \frac{qc_0^2}{\omega}\frac{\partial A}{\partial X_1} = 0 \Rightarrow \frac{\partial A}{\partial T_1} + v_g\frac{\partial A}{\partial X_1} = 0 \Rightarrow A(X_1, T_1, X_2, T_2, \dots) = A\left(X_1 - v_g T_1, X_2, T_2, \dots\right)$$

At this orderwe get the solution $\phi_1 = 0$ or $\phi_1 \propto \phi_0$, thus no new terms add to the general solution

Nonlinear waves in the pendulum chain (IV)

Introducing the notation $D_i = \partial/\partial T_i$ and $D_{X_i} = \partial/\partial X_i$ we get

$$\frac{\partial^2}{\partial t^2} = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots)^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0D_2) + \cdots,$$

$$\frac{\partial^2}{\partial x^2} = \left(D_{X_0} + \varepsilon D_{X_1} + \varepsilon^2 D_{X_2} + \cdots\right)^2 = D_{X_0}^2 + 2\varepsilon D_{X_0} D_{X_1} + \varepsilon^2\left(D_{X_1}^2 + 2D_{X_0}D_{X_2}\right) + \cdots.$$

Inserting the expansion $\theta = \varepsilon\phi_0 + \varepsilon^2\phi_1 + \varepsilon^3\phi_2 + \cdots$ into the SG equation $\theta_{tt} - c_0^2\theta_{xx} + \omega_0^2\theta - \omega_0^2\theta^3/6 = 0$ we get:

- **At order ε^3**:

$$\hat{L}\phi_2 = -D_1^2\phi_0 - 2D_0D_2\phi_0 + c_0^2 D_{X_1}^2\phi_0 + 2c_0^2 D_{X_0}D_{X_2}\phi_0 + \omega_0^2/6\,\phi_0^3 - 2D_0D_1\phi_1 + c_0^2 D_{X_0}D_{X_1}\phi_1$$

We found another driven linear equation whose terms $e^{i\sigma}$ lead to divergence: to avoid it we impose the **solvability condition**

$$-\frac{\partial^2 A}{\partial T_1^2} + 2i\omega\frac{\partial A}{\partial T_2} + c_0^2\frac{\partial^2 A}{\partial X_1^2} + 2iqc_0^2\frac{\partial A}{\partial X_2} + \frac{3}{6}\omega_0^2|A|^2 A = 0$$

Moving to a frame at velocity v_g with the variables $\xi_i = X_i - v_g T_i$ and $\tau_i = T_i$ getting $\frac{\partial A}{\partial T_i} = \frac{\partial A}{\partial \tau_i} - v_g\frac{\partial A}{\partial \xi_i}$ and $\frac{\partial A}{\partial X_i} = \frac{\partial A}{\partial \xi_i}$, thus

$$(c_0^2 - v_g^2)\frac{\partial^2 A}{\partial \xi_1^2} + 2i\omega\left(\frac{\partial A}{\partial \tau_2} - v_g\frac{\partial A}{\partial \xi_2}\right) + 2iqc_0^2\frac{\partial A}{\partial \xi_2} + \frac{1}{2}\omega_0^2|A|^2 A = 0$$

$$i\frac{\partial A}{\partial \tau_2} + \frac{c_0^2 - v_g^2}{2\omega}\frac{\partial^2 A}{\partial \xi_1^2} + \frac{\omega_0^2}{4\omega}|A|^2 A = 0$$

We got the **nonlinear Schrödinger equation (NLSE)** for the envelope A

The nonlinear Schrödinger equation (I)

Let us rewrite the **nonlinear Schrödinger equation (NLSE)** in its usual form

$$i\frac{\partial\psi}{\partial t}+P\frac{\partial^2\psi}{\partial x^2}+Q|\psi|^2\psi=0$$

where $P>0$ and Q depend on the particular studied problem. It's the potential term $-Q|\psi|^2$ that gives the denomination nonlinear to this equation and we will see that for $\boldsymbol{Q>0}$, the solution $\boldsymbol{\psi}$ **is localised** such that it 'digs' its own potential well.

Let us look for **solutions** of the form

$$\psi=\phi(x,t)e^{i\theta(x,t)}$$

where we suppose that both the carrier wave θ and the envelope ϕ are **permanent profile solutions** with different velocities

$$\phi(x,t)=\phi(x-u_e t)\quad \text{and} \quad \theta(x,t)=\theta\left(x-u_p t\right).$$

Inserting the solutions into the NLSE and separating the real and imaginary part, we get:

$$\begin{matrix}-\phi\theta_t+P\phi_{xx}-P\phi\theta_x^2+Q\phi^3=0\\ \phi_t+P\phi\theta_{xx}+2P\phi_x\theta_x=0\end{matrix}\Rightarrow\begin{matrix}u_p\phi\theta_x+P\phi_{xx}-P\phi\theta_x^2+Q\phi^3=0 & (1)\\ -u_e\phi_x+P\phi\theta_{xx}+2P\phi_x\theta_x=0 & (2)\end{matrix}$$

Multiplying (2) by ϕ and integrating in x gives

$$-\frac{u_e}{2}\phi^2+P\phi^2\theta_x=C$$

Since we are focusing to spatially localized solutions, we impose the boundary conditions $\lim_{|x|\to\infty}\phi=\lim_{|x|\to\infty}\phi_x=0$ getting $C=0$

$$\theta_x=\frac{u_e}{2P}$$

The nonlinear Schrödinger equation (II)

$$\theta = \frac{u_e}{2P}(x - u_p t) + C'$$

Integrating

Inserting the latter into (1) we get

$$\frac{u_e u_p}{2P}\phi + P\phi_{xx} - \frac{u_e^2}{4P}\phi + Q\phi^3 = 0$$

Multiplying by $P\phi_x$ and integrating in x

$$\frac{P^2}{2}\phi_x^2 + \frac{PQ}{4}\phi^4 - \frac{u_e^2 - 2u_e u_p}{8}\phi^2 = 0$$

Recognizing $V_{eff}(\phi)$

$$\frac{P^2}{2}\phi_x^2 + V_{eff}(\phi) = 0$$

Integrating with the change of variable $\phi = \phi_0 \operatorname{sech} v$

$$\phi = \phi_0 \operatorname{sech}\left[\sqrt{\frac{Q}{2P}}\phi_0(x - u_e t) + \operatorname{arcsech}\frac{\phi(0,0)}{\phi_0}\right]$$

Putting the center of the soliton $\phi = \phi_0$ at $x = 0$ when $t = 0$

$$\psi(x,t) = \phi_0 \operatorname{sech}\left[\sqrt{\frac{Q}{2P}}\phi_0(x - u_e t)\right] e^{i\frac{u_e}{2P}(x - u_p t)}$$

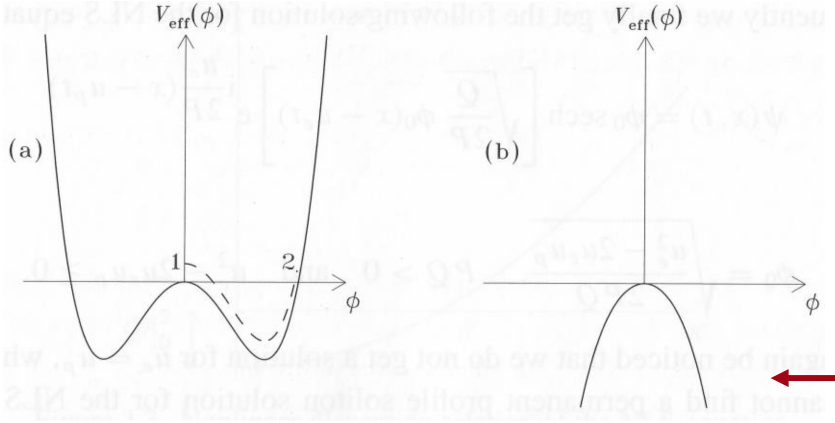
$$\psi = \phi_0 \operatorname{sech}\left(\frac{x - u_e t}{L_e}\right) e^{i(\kappa x - \mu t)}$$

$$L_e = \frac{1}{\phi_0}\sqrt{\frac{2P}{Q}}, \quad \kappa = \frac{u_e}{2P}, \quad \mu = \frac{u_e u_p}{2P}$$

Soliton solution

The nonlinear Schrödinger equation (II)

Shape of the effective potential $V_{eff}(\phi)$ for $PQ > 0$ (a) and $PQ < 0$ (b). [1]



$$V_{eff}(\phi) = \frac{PQ}{4}\phi^4 - \frac{u_e^2 - 2u_e u_p}{8}\phi^2$$

- $\phi \in \mathbb{R} \Rightarrow \phi_x^2 \geq 0 \Rightarrow V_{eff}(\phi) \leq 0$
- $\lim_{\phi \to 0} V_{eff}(\phi) \propto \phi^2 \Rightarrow$
  $u_e^2 - 2u_e u_p \geq 0 \Rightarrow u_e \neq u_p$
  no permanent profile solution
- Bounded motion for $\boldsymbol{PQ > 0}$
- $\phi_0 = \sqrt{(u_e^2 - 2u_e u_p)/2PQ}$

$$\theta = \frac{u_e}{2P}(x - u_p t) + C'$$

Integrating

Inserting the latter into (1) we get

$$\frac{u_e u_p}{2P}\phi + P\phi_{xx} - \frac{u_e^2}{4P}\phi + Q\phi^3 = 0$$

Multiplying by $P\phi_x$ and integrating in x

$$\frac{P^2}{2}\phi_x^2 + \frac{PQ}{4}\phi^4 - \frac{u_e^2 - 2u_e u_p}{8}\phi^2 = 0$$

Recognizing $V_{eff}(\phi)$

$$\frac{P^2}{2}\phi_x^2 + V_{eff}(\phi) = 0$$

Integrating with the change of variable $\phi = \phi_0 \operatorname{sech} v$

$$\phi = \phi_0 \operatorname{sech}\left[\sqrt{\frac{Q}{2P}}\phi_0(x - u_e t) + \operatorname{arcsech}\frac{\phi(0,0)}{\phi_0}\right]$$

Putting the center of the soliton $\phi = \phi_0$ at $x = 0$ when $t = 0$

$$\psi(x,t) = \phi_0 \operatorname{sech}\left[\sqrt{\frac{Q}{2P}}\phi_0(x - u_e t)\right] e^{i\frac{u_e}{2P}(x - u_p t)}$$

$$\psi = \phi_0 \operatorname{sech}\left(\frac{x - u_e t}{L_e}\right) e^{i(\kappa x - \mu t)}$$

$$L_e = \frac{1}{\phi_0}\sqrt{\frac{2P}{Q}}, \quad \kappa = \frac{u_e}{2P}, \quad \mu = \frac{u_e u_p}{2P}$$

Soliton solution

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).

Solitons in optical fibers (I)

Optical fiber communication is one of the **main applications** of solitons

When a signal propagates in an optical fibre, **nonlinear effects become important** because:

- small cross section $\sim 10^{-6}\ \mathrm{cm}^2 \Rightarrow$ **high power** densities $\sim \mathrm{MW/cm^2}$;
- long distances makes nonlinear terms **no longer negligible**.

Optical fibers are made of an isotropic medium $\Rightarrow\ \chi^{(2)} = 0$ and $\chi^{(3)} \neq 0$ where the THG contribute is negligible wrt OKE

$$\vec{P}(\vec{r},t) = \varepsilon_0 \chi^{(1)}(\omega)\overrightarrow{E_0}(r)e^{-i(\vec{k}\cdot\vec{r}-\omega t)} + \varepsilon_0 \chi^{(3)}(\omega)\overrightarrow{E_0}\left|\overrightarrow{E_0}\right|^2 e^{-i(\vec{k}\cdot\vec{r}-\omega t)}$$

To find the structure of the wave in the fiber, we start from Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{and} \quad \vec{\nabla} \times \vec{H} = -\frac{\partial \vec{D}}{\partial t}$$

that together give the wave equation

$$\nabla^2 \vec{E} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \vec{D}}{\partial t^2} = 0$$

Focusing on the propagation in the **transverse direction** of the fiber, thus over small distances, it is sufficient to consider the **linear part of the polarization** into $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi^{(1)} \vec{E} = \varepsilon(\omega)\vec{E}(\vec{r},t)$ giving

$$\nabla^2 \vec{E}(\vec{r},\omega) + \frac{\omega^2}{\varepsilon_0 c^2}\varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega) = 0$$

with **guided wave solutions** and dispersion relation given by

$$\vec{E}(\vec{r},\omega) = \vec{U}(\vec{r}_\perp,\omega)e^{i(kz-\omega t)}, \qquad k_n = \frac{\omega \beta_n(\omega)}{c}$$

Solitons in optical fibers (II)

Focusing now on the propagation in the **longitudinal direction** of the fiber, we must account for nonlinearity

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\vec{E} + \varepsilon_0\chi^{(1)}\vec{E} + \varepsilon_0\chi^{(3)}\left|\vec{E}\right|^2\vec{E} = \varepsilon(\vec{r},\omega)\vec{E}(\vec{r},\omega) + \varepsilon_0\chi^{(3)}\left|\vec{E}\right|^2\vec{E} = \vec{D}_l(\vec{r},\omega) + \varepsilon_0\chi^{(3)}\left|\vec{E}\right|^2\vec{E}$$

Thus the wave equation becomes

$$\nabla^2\vec{E} - \frac{1}{\varepsilon_0 c^2}\frac{\partial^2\vec{D}}{\partial t^2} = \frac{\chi^{(3)}}{c^2}\frac{\partial^2\left|\vec{E}\right|^2\vec{E}}{\partial t^2} \qquad (3)$$

in which we want to consider a solution of the form

$$\vec{E}(\vec{r},t) = \phi(z,t)\vec{U}(\vec{r}_\perp,\omega_0)e^{i(k_0 z-\omega_0 t)} + c.c.$$

The right hand side of (3) gives, with $|U|^2 = 1$:

$$\frac{\chi^{(3)}}{c^2}\frac{\partial^2\left|\vec{E}\right|^2\vec{E}}{\partial t^2} \cong -\frac{\omega_0^2}{c^2}\chi^{(3)}|\phi|^2\phi U(0,\omega_0)e^{i(k_0 z-\omega_0 t)} \qquad (4)$$

The first term of the left hand side gives:

$$\nabla^2\vec{E} = \nabla^2\phi U e^{i(k_0 z-\omega_0 t)} + 2\vec{\nabla}\phi\vec{\nabla}U e^{i(k_0 z-\omega_0 t)} + \phi\nabla^2\left(U e^{i(k_0 z-\omega_0 t)}\right)$$
$$= \frac{\partial^2\phi}{\partial z^2}U e^{i(k_0 z-\omega_0 t)} + 2i\frac{\partial\phi}{\partial z}k_0 U e^{i(k_0 z-\omega_0 t)} + \phi\nabla^2\left(U e^{i(k_0 z-\omega_0 t)}\right) \qquad (5)$$

While the second term of the left hand side can be calculated in Fourier space expanding to the second order $\varepsilon(\omega)$ around ω_0:

$$\frac{\partial^2 D(z,t)}{\partial t^2} = \left[-\omega_0^2\varepsilon(\omega_0)\phi - 2i\omega_0\left(\varepsilon + \frac{\omega_0}{2}\frac{\partial\varepsilon}{\partial\omega}\right)\frac{\partial\phi}{\partial t} + \left(\varepsilon + 2\omega_0\frac{\partial\varepsilon}{\partial\omega} + \frac{\omega_0^2}{2}\frac{\partial^2\varepsilon}{\partial\omega^2}\right)\frac{\partial^2\phi}{\partial t^2}\right]U(\omega_0)e^{i(k_0 z-\omega_0 t)} \qquad (6)$$

Solitons in optical fibers (III)

We now insert (4), (5) and (6) into (3) simplifyng the factor $Ue^{i(k_0 z-\omega_0 t)}$ and noticing that the prefactor of the ϕ term vanishes. We get the equation determining the time evolution of the envelope:

$$\frac{\partial^2\phi}{\partial z^2}+2ik_0\frac{\partial\phi}{\partial z}+\underbrace{\frac{2i\omega_0}{\varepsilon_0 c^2}\left(\varepsilon+\frac{\omega_0}{2}\frac{\partial\varepsilon}{\partial\omega}\right)}\frac{\partial\phi}{\partial t}-\underbrace{\frac{1}{\varepsilon_0 c^2}\left(\varepsilon+2\omega_0\frac{\partial\varepsilon}{\partial\omega}+\frac{\omega_0^2}{2}\frac{\partial^2\varepsilon}{\partial\omega^2}\right)}\frac{\partial^2\phi}{\partial t^2}+\frac{\omega_0^2}{c^2}\chi^{(3)}|\phi|^2\phi=0$$

$$k^2=\frac{\omega^2}{c_n^2}=\frac{\omega^2\varepsilon(\omega)}{c^2\varepsilon_0}\longrightarrow\frac{\partial k}{\partial\omega}=\frac{1}{v_g}=\frac{\omega\varepsilon(\omega)}{kc^2\varepsilon_0}+\frac{1}{2k}\frac{\omega^2}{c^2\varepsilon_0}\frac{\partial\varepsilon}{\partial\omega}\longrightarrow\frac{\partial}{\partial\omega}\left(\frac{k}{v_g}\right)=\frac{1}{v_g}\frac{\partial k}{\partial\omega}+k\frac{\partial}{\partial\omega}\left(\frac{1}{v_g}\right)=\frac{\varepsilon(\omega)}{c^2\varepsilon_0}+\frac{2\omega}{c^2\varepsilon_0}\frac{\partial\varepsilon}{\partial\omega}+\frac{\omega^2}{2c^2\varepsilon_0}\frac{\partial^2\varepsilon}{\partial\omega^2}$$

Evaluating in $\omega=\omega_0$ the equation becomes

$$\frac{\partial^2\phi}{\partial z^2}+2ik_0\left[\frac{\partial\phi}{\partial z}+\frac{1}{v_g}\frac{\partial\phi}{\partial t}\right]-k_0\frac{\partial}{\partial\omega}\left(\frac{1}{v_g}\right)\frac{\partial^2\phi}{\partial t^2}-\frac{1}{v_g^2}\frac{\partial^2\phi}{\partial t^2}+\frac{\omega_0^2}{c^2}\chi^{(3)}|\phi|^2\phi=0$$

which for the frame change $\tau=t-z/v_g$ and $\xi=z$ becomes the **NLSE**

$$i\frac{\partial\phi}{\partial\xi}-\frac{1}{2}\frac{\partial}{\partial\omega}\left(\frac{1}{v_g}\right)\frac{\partial^2\phi}{\partial\tau^2}+\frac{\omega_0^2}{2k_0c^2}\chi^{(3)}|\phi|^2\phi=0$$

with **soliton solutions**

$$\phi=\phi_0\,\mathrm{sech}\left[\sqrt{\frac{Q}{2P}}\phi_0\xi\right]e^{i\frac{Q\phi_0^2}{2}\xi}$$

Solitons in optical fibers (IV)

The NLSE for an optical fiber was proposed in **1973** by **Hasegawa** and **Tappert** [5,6]

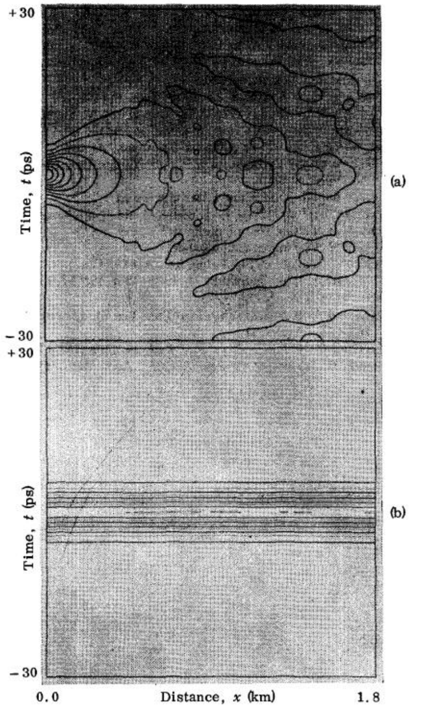


FIG. 1. Comparison of linear (a) and stationary nonlinear (b) propagation of 3-ps optical pulses in glass fibers.

[5]

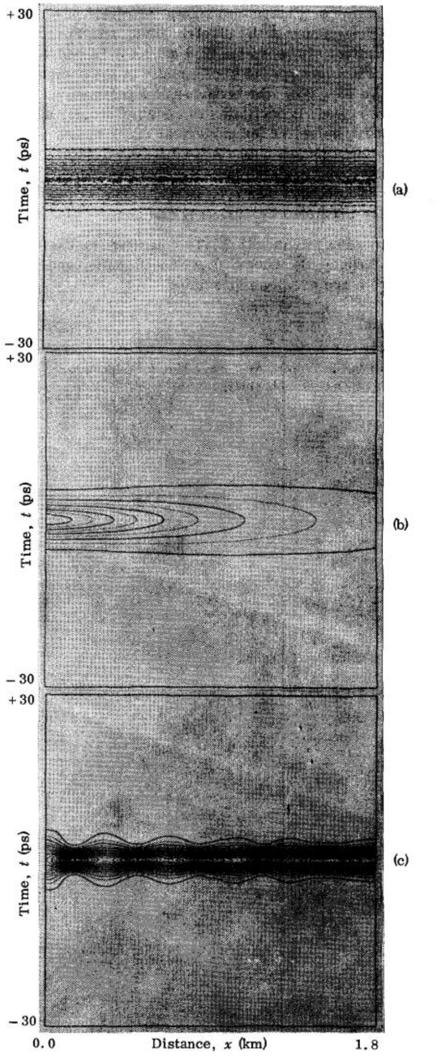


FIG. 2. Stability of stationary nonlinear pulses under the actions of (a) noise, (b) absorption, and (c) large perturbation.

[5]

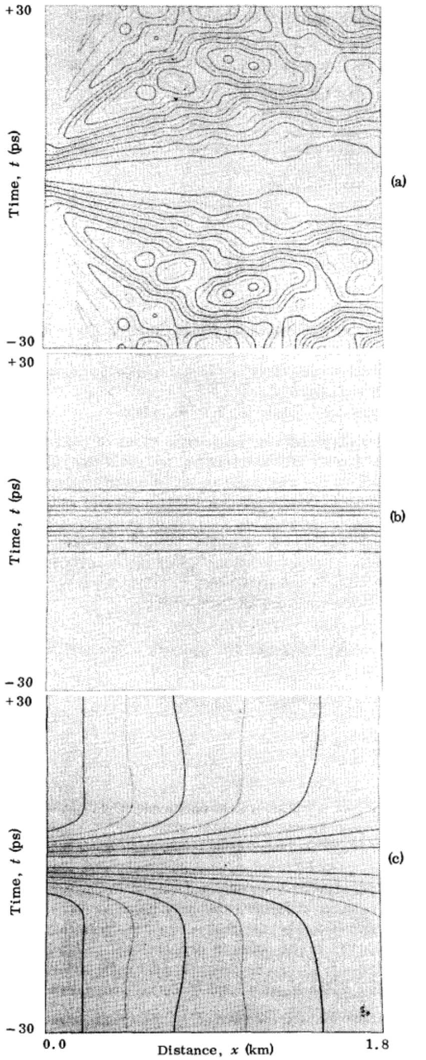


FIG. 1. Comparison of (a) linear dark pulse, (b) stationary nonlinear dark pulse, and (c) nonlinear dark pulse with absorption.

[6]

[5] Hasegawa, A., Tappert, F., *Appl. Phys. Lett.* **23.3**, 142-144 (1973).
[6] Hasegawa, A., Tappert, F., *Appl. Phys. Lett.* **23**, 171-2 (1973).

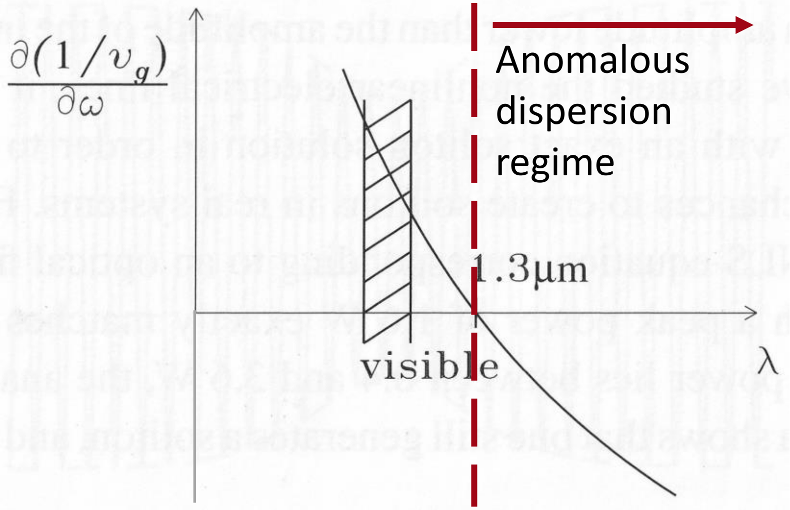
Solitons in optical fibers (IV)

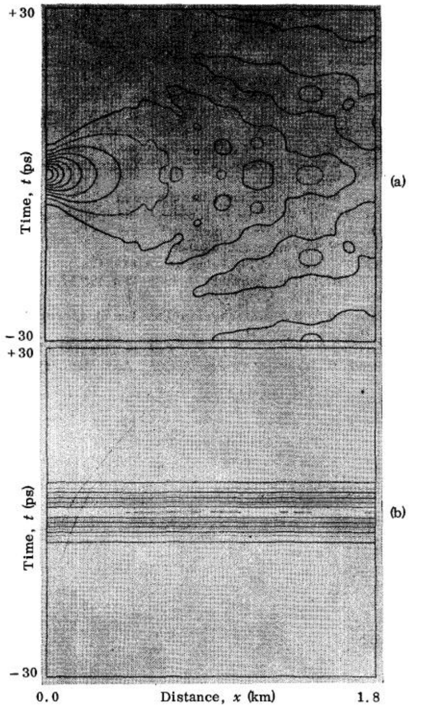
The NLSE for an optical fiber was proposed in **1973** by **Hasegawa** and **Tappert** [5,6]

First experimental checks after two technical problems have been overcome:

- **availability** of mono-mode, thin, low-loss fibers;
- get the **condition $PQ > 0$** where NLSE has soliton solutions:
  - $\chi^{(3)} > 0 \Rightarrow Q > 0$ always;
  - to get $P > 0$ we need $\frac{\partial}{\partial\omega}\left(\frac{1}{v_g}\right) < 0$

Schematic plot of the group velocity dispersion versus the wavelength for a silica fiber. [1]





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[5]

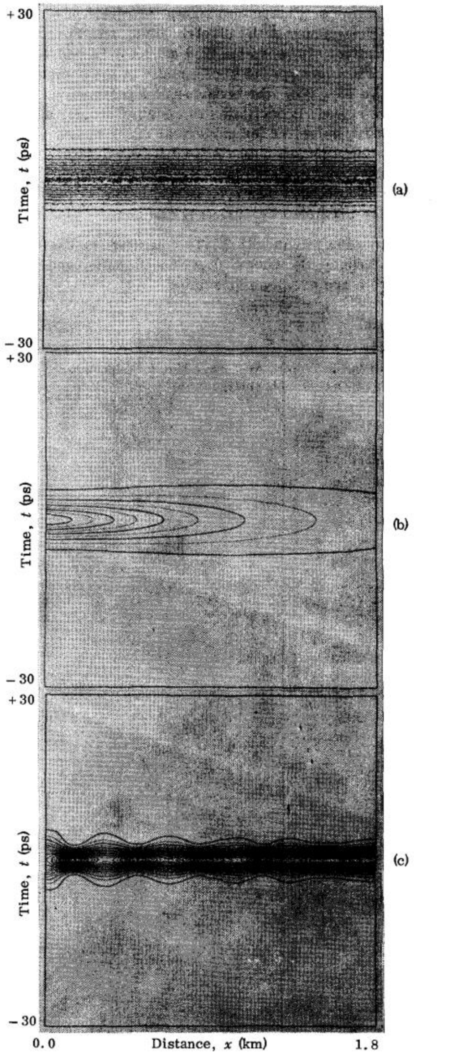


FIG. 2. Stability of stationary nonlinear pulses under the actions of (a) noise, (b) absorption, and (c) large perturbation.

[5]

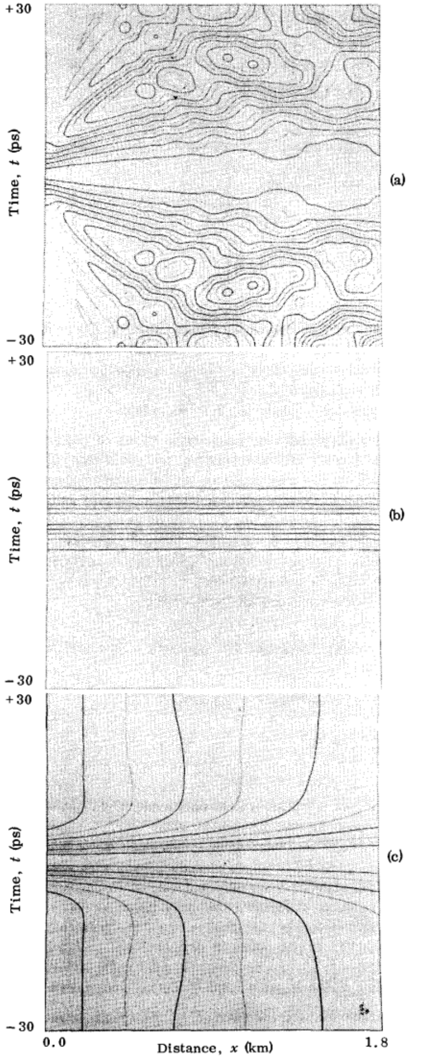


FIG. 1. Comparison of (a) linear dark pulse, (b) stationary nonlinear dark pulse, and (c) nonlinear dark pulse with absorption.

[6]

[1] Dauxois, T. and Peyrard, M., Cambridge University Press (2006).
[5] Hasegawa, A., Tappert, F., *Appl. Phys. Lett.* **23.3**, 142-144 (1973).
[6] Hasegawa, A., Tappert, F., *Appl. Phys. Lett.* **23**, 171-2 (1973).

Solitons in optical fibers (V)

In **1980**, **Mollenauer, Stolen** and **Islam** reported the first **experimental observation** in optical fibers of **pulse compression** and **pulse splitting**, that at certain critical power levels is characteristic of higher-order **solitons**:

- single-mode silica-glass **fibers** (low losses of $0.2\ \mathrm{dB/km}$ at $\lambda \sim 1.3\ \mu\mathrm{m}$);
- mode-locked color-center **laser** (F_2^+ centers in NaCl);
- **autocorrelation** measurement of laser and fiber output;
- **power** dependent trend:
  - $P < 0.3\ \mathrm{W}$ broadening due to dispersion
  - $P < 1.2\ \mathrm{W}$ narrowing up to the recovery of the input width at $P = 1.2\ \mathrm{W}$
  - $P = 5\ \mathrm{W}$ reached narrowest width (2 ps)
  - Broad base rises and splits up to $P = 11.4\ \mathrm{W}$ with the first well resolved splitting (three-fold)
  - $P = 22.5\ \mathrm{W}$ five-fold splitting
- Computed NLSE solutions confirm that the observed autocorrelation traces closely agree with the behavior of $\boldsymbol{N = 1, 2, 3, 4}$ **solitons**;
- The average power is constant: $\boldsymbol{P_0} = P/N^2 = \mathbf{1.24\ W}$.

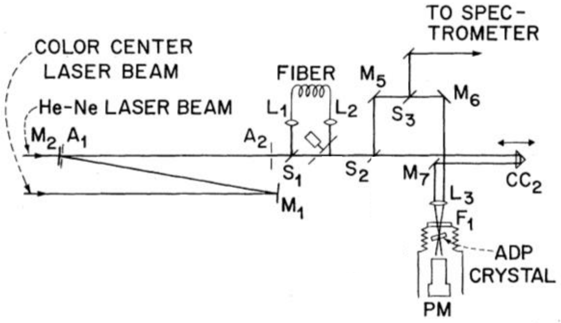


FIG. 1. Schematic of the apparatus. M_1, M_2, A_1, and A_2 constitute a simple beam-aiming device. At S_1, the beam is split between "fiber" and "laser" channels. The chopper alternately blocks the two beams before they enter the autocorrelator (S_2, M_5, M_6, M_7, CC_2, etc.); the resultant photomultiplier signals (from noncollinear second harmonic generation in the ammonium dihydrogen phosphate crystal) are then seperated out electronically. F_1, a slab of Si, passes 1.55-μm light and rejects room light.

[7]

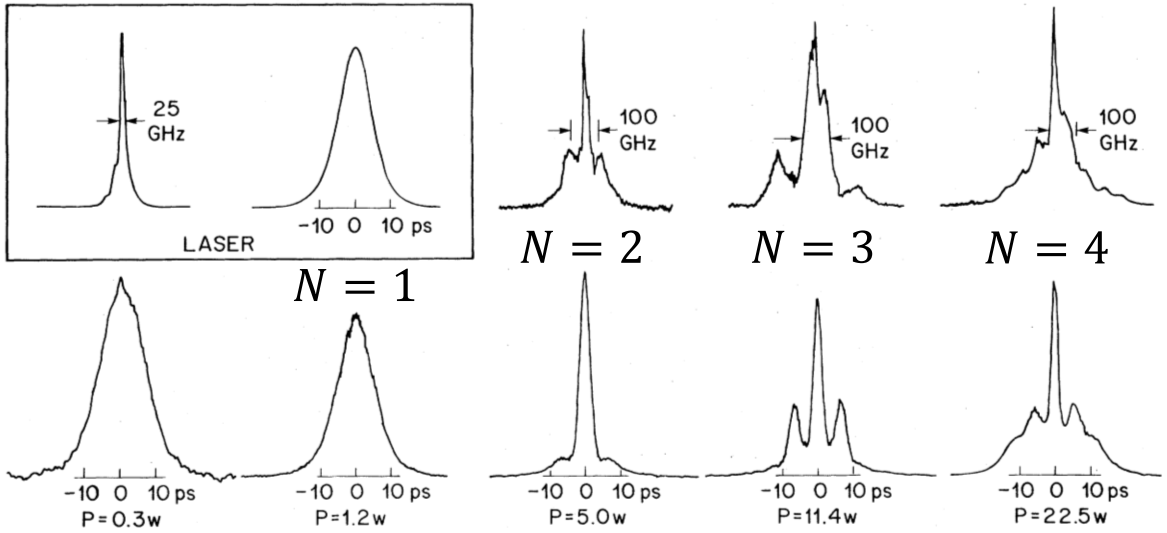


FIG. 2. Below: Autocorrelation traces of the fiber output as a function of power. Above: Corresponding frequency spectra. Inset: Similar data for the direct laser output. There is no absolute intensity scale here; the various curves have been roughly normalized to a common height. Corresponding to the fiber data, from low to high power, the laser pulse widths were 7.2, 7.0, 6.1, 6.8, and 6.2 ps, respectively. See text.

[7] Mollenauer, L.F., Stolen, R.H., Islam, M.N., *Phys. Rev. Lett.* **45**, 1095-8 (1980).

Solitons in optical fibers (VI)

- In **1974**, **Satsuma** and **Yajima** [8] studied the **time evolution** of the single soliton initial condition

$$\phi = a\phi_0 \operatorname{sech}\left[\sqrt{\frac{Q}{2P}}\phi_0\tau\right]$$

that resulted to give rise to a soliton for $a \in [0.5,\ 1.5]$, that for values of an optical fiber corresponds to $P \in [0.4\mathrm{W}, 3.6\mathrm{W}]$ thus demonstrating the **exceptional stability** of solitons to distrortions.
- Obtaining **multisolitons** is a more stringent test of validity for NLSE description: N is the largest integer such that $N \leq a + \frac{1}{2}$
- **Temporal** and **spectral restoration** of optical pulses is observed from a fiber with **length equal to the soliton period** at **powers** corresponding to **integral multiples of the amplitude** of the fundamental soliton [9].
- **More solitons** are in the solution, sharper is the pulse: from 7 ps input pulse to sub-ps output [10]

[8]

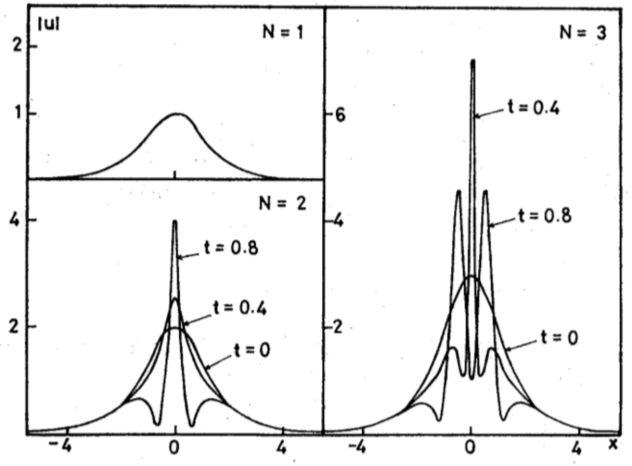
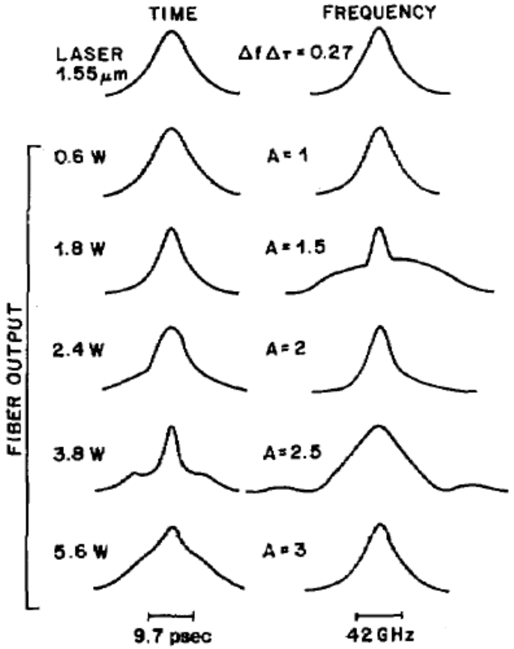
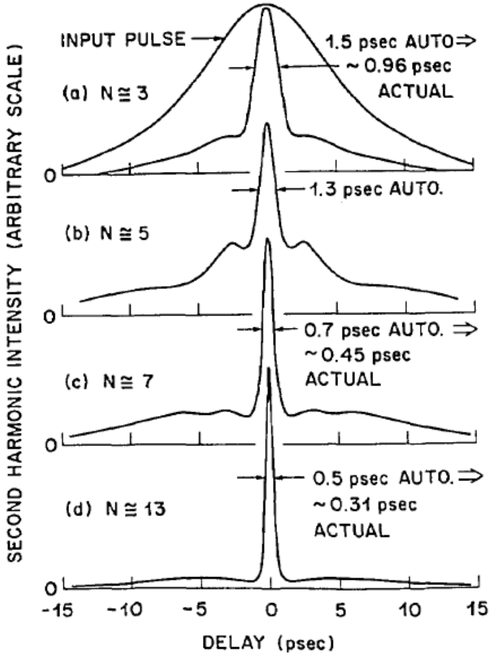


Fig. 1. Exact solutions for $u(x, t=0) = N \operatorname{sech} x$.



Experimental pulse shapes (autocorrelation) and spectra. [9]



Autocorrelation traces of pulses from a 320m long fiber. [10]

[8] Satsuma, J., Yajima, N., *Progress of Th. Phys. Suppl.* **55**, 284-306 (1974).
[9] Stolen, R. H., Mollenauer, L. F., Tomlinson, W. J., *Opt. Lett.* **8.3**, 186-188 (1983).
[10] Mollenauer, L. F., Stolen, R. H., Gordon, J. P., Tomlinson, W. J., *Opt. Lett.* **8.5**, 289-291 (1983).

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[10] Mollenauer, L. F., Stolen, R. H., Gordon, J. P., Tomlinson, W. J., Extreme picosecond pulse narrowing by means of soliton effect in single-mode optical fibers, *Optics Letters* **8.5**, 289-291 (1983).

Thank you for the attention!