Josephson Junctions

Course:
'Quantum physics with atoms and ions' by Prof. Luca Salasnich

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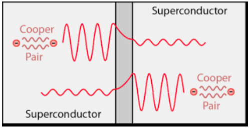
Superconductive Josephson junction

Introduction

- Theorized by Brian Josephson in 1962, experimentally realized the following year.
- Josephson junctions consist of two or more **superconductors** coupled by a **weak link**.
- Characterized by the quantum tunneling of **superconductive Cooper pairs**, which are pairs of electrons with total spin zero.
- Employed in a several cutting-edge applications such as
  - Superconducting quantum computers (Qubits)
  - High-sensitivity devices (SQUIDs)
  - Quantum simulations

Nobel Prize (1973).
"For his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena which are generally known as the Josephson effect"



Scheme of a superconductive Josephson junction

Josephson Equations - I

The gas of **Cooper pairs** can be described by a macroscopic wavefunction

$$\Psi(\vec{r},t) = \underbrace{\sqrt{N_L(t)}e^{i\varphi_L(t)}}_{\psi_L(t)}\phi_L(\vec{r}) + \underbrace{\sqrt{N_R(t)}e^{i\varphi_R(t)}}_{\psi_R(t)}\phi_R(\vec{r}) \tag{1}$$

whose time-dependent parts obey the following coupled equations

$$\begin{cases} i\hbar\partial_t\psi_L(t) = U_L\psi_L(t) + K\psi_R(t) \\ i\hbar\partial_t\psi_R(t) = U_R\psi_R(t) + K\psi_L(t) \end{cases} \tag{2}$$

where $U_L = -U_R = eV$ depends on the electrical charge e and the external potential V, while K is the coupling energy (tunneling energy).

Josephson Equations - II

Subtracting the first equation the second one in Eq.(2) and separeting the real part from the imaginary part, one can obtain the **Josephson equations**[2]. These equations are given by

$$\begin{cases} I(t) &= I_c \sin\varphi(t) \\ \frac{d\varphi}{dt} &= \frac{2e}{\hbar} V \end{cases} \tag{3}$$

where $\varphi(t) = \varphi_L(t) - \varphi_R(t)$ is the phase difference, which is called **Josephson phase**, I_C is the critical current, defined as

$$I_c \equiv \frac{4}{\hbar} Ke\sqrt{N_L(t)N_R(t)} \simeq \frac{2}{\hbar} KeN \tag{4}$$

since in superconductors $N_L(t) \simeq N_R(t) \simeq N/2$ and the electric current is given by

$$I = e[\dot{N}_L(t) - \dot{N}_R(t)]. \tag{5}$$

[2] B. D. Josephson, Physics Letters **1.7** (1962)

Josephson effects

Josephson predicted two main effects from the Josephson equations, namely:

- **DC Josephson effect**: In the absence of an external potential V, there is still a direct current I proportional to the sine of the Josephson phase $\varphi(0)$:

$$V = 0 \implies I = I_c \sin \varphi(0) \tag{6}$$

- **AC Josephson effect**: In the presence of a constant external potential V, there is a sinusoidal alternating current:

$$V = V_0 \implies I(t) = I_c \sin\left(\frac{2eV_0 t}{\hbar} + \varphi(0)\right) \tag{7}$$

Electronics in Josephson Junction

Josephson junction are interesting element in electronics due to the non-linear relation between the electric current and the external potential. Integrating in time the second Josephson equation we obtain the general expression for the Josephson phase

$$\varphi(t) = \varphi(0) + \frac{2e}{\hbar}\int_0^t V(t')dt' \tag{8}$$

and inserting this expression into the first Josephson equation the general expression for the electric current is

$$I(t) = I_C \sin\left(\varphi(0) + \frac{2e}{\hbar}\int_0^t V(t')dt'\right) \tag{9}$$

RCSJ model

The **Resistively Capacitance Shunted Junction** (RCSJ) model[3] accounts the fact that real Josephson Junction have resistive and capacitive effects, the total current is given by

$$
\begin{aligned}
I &= I_{Jos} + I_{Res} + I_{Cap} \\
&= I_c \sin\varphi + \frac{\hbar}{2eR}\dot{\varphi} + \frac{\hbar C}{2e}\ddot{\varphi}
\end{aligned}
\tag{10}
$$

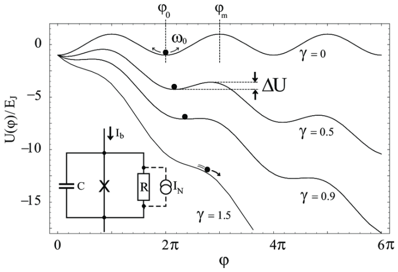
where I_{Jos}, I_{Res} and I_{Cap} are the Josephson, resistive and capacitive terms of the current.

[3] D. E. McCumber, J. Appl. Phys. **39** (1968)

RCSJ model

We can rewrite Eq.(10) as an equation of motion for the Josephson phase φ:

$$M\ddot{\varphi} + \eta\dot{\varphi} + \frac{\partial U}{\partial \varphi} = 0 \quad (11)$$



Washboard potential in RCSJ model

which is equivalent to the motion of a particle of mass $M = C\frac{\hbar^2}{4e^2}$ and damping $\eta = \frac{\hbar^2}{4e^2R}$ in a washboard potential $U(\varphi) = -E_J \cos\varphi - \gamma\varphi$, where $E_J = \frac{\hbar I_c}{2e}$ and $\gamma = \frac{\hbar I}{2e}$. From this model one can construct the **phase qubit**, a device designed to operate as a quantum bit.

Atomic Josephson junction

Atomic Josephson Junction - Weak Link Assumpiton

The atomic Josephson junction consists of an ultracold bosonic gas in a **symmetric** double well potential U_{DW}. The potential is such that its energy levels are **doublets** of quasi-degenerate **single-particle energy levels**.
Hence, we can expand the single-particle wavefunction as

$$\phi(x,t) = f_0(x)\psi_0(t) + f_1(x)\psi_1(t) + f_2(x)\psi_2(t) + \dots \tag{12}$$

Under the **weak link assumption** only the first doublet (the single-particle ground state ε_0 and the single-particle first-excited state ε_1) are occupied, the single-particle wavefunction is approximated as

$$\phi(x,t) \simeq f_0(x)\psi_0(t) + f_1(x)\psi_1(t) \tag{13}$$

Due to the qualitative shape of $f_0(x)$ and $f_1(x)$ one can define two functions

$$\varphi_{L,R}(x) \equiv \frac{f_0(x) \mp f_1(x)}{\sqrt{2}} \tag{14}$$

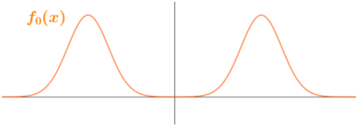
which let us use the **two-mode approximation**, rewriting the single-particle wavefunction

$$\phi(x,t) \simeq \varphi_L(x)\psi_L(t) + \varphi_R(x)\psi_R(t) \tag{15}$$

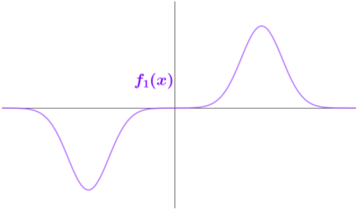
in which the space and the time dependence are separated. The single-particle wavefunction obeys the 1DGPE:

$$i\hbar\frac{\partial}{\partial t}\phi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U_{DW}(x) + g|\phi(x,t)|^2\right]\phi(x,t) \tag{16}$$

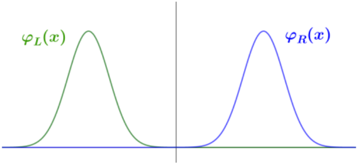
Atomic Josephson Junction - Two-mode approximation



(a) Qualitative shape of $f_0(x)$



(b) Qualitative shape of $f_1(x)$



(c) Qualitative shapes of $\varphi_L(x)$ and $\varphi_R(x)$

Atomic Josephson junction - Madelung Transformations

Taking the **Madelung transformation** for the time-dependent part, it is possible to express them as a function of the number of particles in the left/right side of the box, $N_{L,R}(t)$ and their phase $\theta_{L,R}(t)$

$$\psi_{L,R}(t) = \sqrt{N_{L,R}(t)}e^{i\theta_{L,R}(t)} \tag{17}$$

Note that the quantity $N = N_L(t) + N_R(t)$ is a constant of motion. Meanwhile, the new **dynamical quantities** are

$$\begin{cases} z(t) = \frac{N_L(t) - N_R(t)}{N} & \text{Fractional population imbalance} \\ \theta(t) = \theta_R(t) - \theta_L(t) & \text{Phase difference} \end{cases} \tag{18}$$

Josephson-Smerzi equations

The Josephson-Smerzi equation[4], i.e. the Josephson equations for a Bose-Einstein condensate, are obtained inserting Eq.(17) and Eq.(18) into Eq.(16)

$$\begin{cases} \dot{z}(t) &= -\frac{2K}{\hbar}\sqrt{1-z^2}\sin\theta \\ \dot{\theta}(t) &= \frac{2K}{\hbar}\frac{z}{\sqrt{1-z^2}}\cos\theta + \frac{N\gamma}{\hbar}z \end{cases} \tag{19}$$

where K is the **tunneling energy**, associated to the tunneling effect

$$K = -\int_{-\infty}^{+\infty} \varphi_L(x)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U_{DW}(x)\right]\varphi_R(x)dx \tag{20}$$

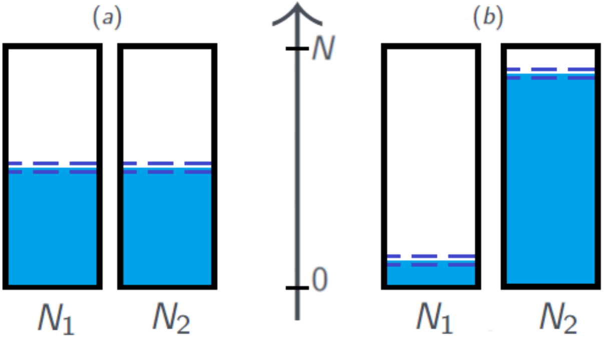
while γ is the **coupling strength** and is linked to the inter-atomic potential

$$\gamma = g\int_{-\infty}^{+\infty} \varphi_{L,R}^4(x)dx. \tag{21}$$

[4] A. Smerzi et al., Phys. Rev. Lett. **79**, 25 (1997)

Josephson Oscillations and MQST

From the Josephson-Smerzi equations is possible to study two interesting phenomena that occurs in the atomic Josephson junction under particular conditions. These phenomena are the Josephson oscillations and the Macroscopic Quantum Self Trapping (MQST).



(a) Josephson oscillations (b) Macroscopic Quantum Self Trapping

Josephson Oscillations

Consider now the case in which the fractional population imbalance and the phase difference are infinitesimal, i.e.
$|z(t)| \ll 1, |\theta(t)| \ll 1$, then the Josephson-Smerzi equation become

$$\begin{cases} \dot{z}(t) & = -\frac{2K}{\hbar}\theta \\ \dot{\theta}(t) & = \left(\frac{2K}{\hbar} + \frac{N\gamma}{\hbar}\right) z \end{cases} \tag{22}$$

and they can be rewritten as the **equation of motion of an harmonic oscillator** for both z and θ

$$\begin{cases} \ddot{z}(t) + \Omega^2 z & = 0 \\ \ddot{\theta}(t) + \Omega^2 \theta & = 0 \end{cases} \tag{23}$$

where $\Omega \equiv \frac{1}{\hbar}\sqrt{4K^2 + 2KN\gamma}$ is the **Josephson frequency**. In the non-interacting limit ($\gamma = 0$), the frequency reduces to the Rabi frequency $\Omega_R = \frac{2K}{\hbar}$.

Macroscopic Quantum Self Trapping

The conserved energy of the system is given by

$$E(z,\theta) = \frac{\gamma N^2}{4} z^2 - KN\sqrt{1-z^2}\cos\theta \qquad (24)$$

To obtain the MQST, the average fractional population imbalance must be different from zero, i.e. $\langle z(t)\rangle \neq 0$, which is equivalent to impose the following constraint on the conserved energy

$$E(z_0,\theta_0) > E(0,\pi) \qquad (25)$$

Defining then the strength parameter $\Lambda \equiv \gamma N/2K$, then the MQST occurs if the strength parameter is higher then a certain critical value Λ_c, which depends on the initial conditions of the system (z_0,θ_0), namely

$$\Lambda > \Lambda_c \equiv \frac{1+\sqrt{1-z_0^2}\cos\theta_0}{z_0^2/2} \qquad (26)$$

To retrieve the superconductive Josephson equation one has to consider an **asymmetric double-well potential**.
Calling ΔU the energy difference between the two local minima the Josephson-Smerzi equations are

$$\begin{cases} \dot{z}(t) & = -\frac{2K}{\hbar}\sqrt{1-z^2}\sin\theta \\ \dot{\theta}(t) & = \frac{2K}{\hbar}\frac{z}{\sqrt{1-z^2}}\cos\theta + \gamma z + \frac{\Delta U}{\hbar} \end{cases} \tag{27}$$

Since in the superconductors the fractional population imbalance is very low, one can consider $|z| \ll 1$, $\gamma|z| \ll 1$ and $K|z|/\hbar \ll 1$ obtaining the Josephson equations

$$\begin{cases} I & = -\dot{z}(t) = \frac{2K}{\hbar}\sin\theta \\ \dot{\theta} & = \frac{\Delta U}{\hbar} \end{cases} \tag{28}$$

Finally, the Josephson current is $I(t) = I_c \sin\left(\theta_0 + \frac{\Delta U}{\hbar}t\right)$, where $I_c = \frac{2K}{\hbar}$ is the critical current.

Conclusion

- We found how to derive the **Josephson equations** for superconductors and the **Josephson-Smerzi equations** for superfluids.
- From the former we analyzed the **Josephson effects** and the **RCSJ model**.
- From the latter we studied the **Josephson Oscillations** and the **Macroscopic Quantum Self Trapping**.
- Finally, we saw that under particular conditions, from the Josephson-Smerzi equations it is possible to obtain the Jospehson ones.