IDEAL FERMI GAS AT ZERO TEMPERATURE :
SEMICLASSICAL DENSITY PROFILE

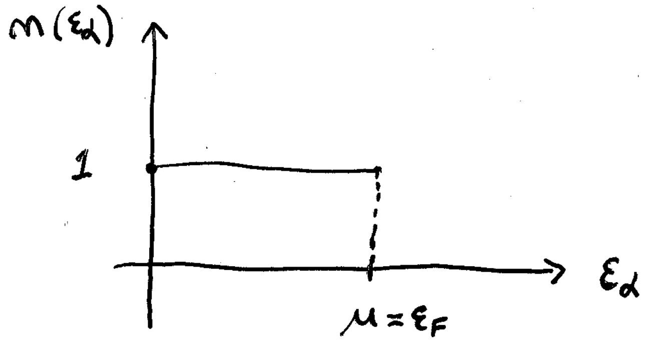
$$n_\alpha = \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)} + 1} \xrightarrow[T \to 0]{} \theta(\mu - \varepsilon_\alpha)$$

$$\beta = \frac{1}{K_B T}$$

$\boxed{T = 0}$

$$N = \sum_\alpha \theta(\mu - \varepsilon_\alpha)$$

$\varepsilon_F = \mu(T=0)$ FERMI ENERGY



$$\varepsilon_\alpha \to \varepsilon(\vec{r}, \vec{p}) \simeq \frac{p^2}{2m} + U(\vec{r})$$

SEMICLASSICAL APPROXIMATION

$$N = \int \frac{d^3\vec{p}\, d^3\vec{r}}{(2\pi\hbar)^3}\, \theta(\mu - \varepsilon(\vec{r}, \vec{p}))$$

$$= \int \frac{dp\, p^2\, 4\pi\, d^3\vec{r}}{(2\pi\hbar)^3}\, \theta\left(\mu - \frac{p^2}{2m} - U(\vec{r})\right)$$

$$= \int d^3\vec{r} \int_0^{\sqrt{2m(\mu - U(\vec{r}))}} \frac{4\pi}{8\pi^3\hbar^3}\, p^2\, dp = \int d^3\vec{r}\, n(\vec{r})$$

$$n(\vec{r}) = \frac{1}{2\pi^2\hbar^3}\left[\frac{1}{3} p^3\right]_0^{\sqrt{2m(\mu - U(\vec{r}))}} = \frac{1}{6\pi^2\hbar^3}\left(\sqrt{2m(\mu - U(\vec{r}))}\right)^3$$

$$\boxed{n(\vec{r}) = \frac{(2m)^{3/2}}{6\pi^2\hbar^3}\left(\mu - U(\vec{r})\right)^{3/2}}$$

DENSITY PROFILE AT $T=0$

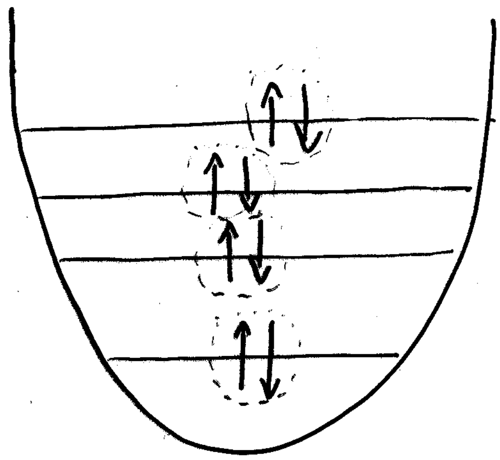
IN THE UNIFORM CASE ($U(\vec{r}) = 0$) :

$$\mu = \varepsilon_F = \frac{\hbar^2 K_F^2}{2m} \quad \text{WITH} \quad K_F = (6\pi^2 n)^{1/3}$$

FERMI ENERGY — FERMI WAVE-NUMBER

SUPERFLUID FERMI GAS
AND MACROSCOPIC WAVE-FUNCTION

FERMI GAS WITH TWO-SPIN COMPONENTS $\quad N_\uparrow = N_\downarrow$

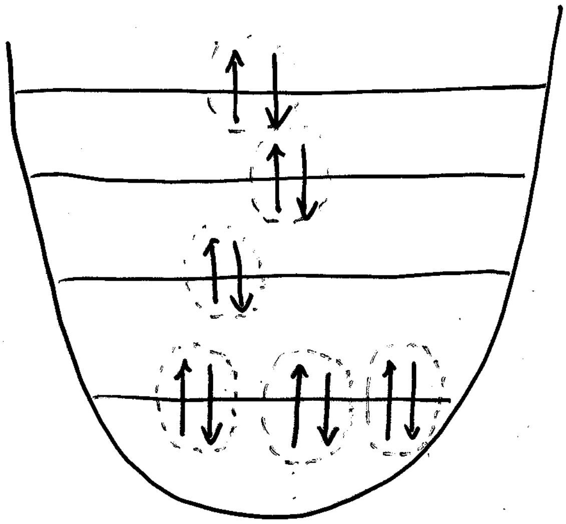


<u>IDEAL FERMI GAS</u>:

ONLY ONE PAIR IN THE LOWEST SINGLE-PARTICLE STATE (PAULI PRINCIPLE)

DUE TO THE <u>ATTRACTIVE</u> INTERACTION:

FORMATION OF PECULIAR PAIRS CALLED "COOPER PAIRS" AND THE FERMI GAS IS CALLED "SUPERFLUID FERMI GAS", OR "BCS FERMI GAS".



<u>ATTRACTIVE FERMI GAS</u>:

SOME PAIRS IN THE LOWEST SINGLE-PARTICLE STATE (AGAINST PAULI PRINCIPLE)

$\Xi(\hat{r})$ IS THE WAVE FUNCTION OF THE LOWEST SINGLE-PARTICLE STATE, USUALLY NORMALIZED TO THE NUMBER $\frac{N_0}{2}$ OF PAIRS IN THIS STATE (CONDENSED PAIRS).

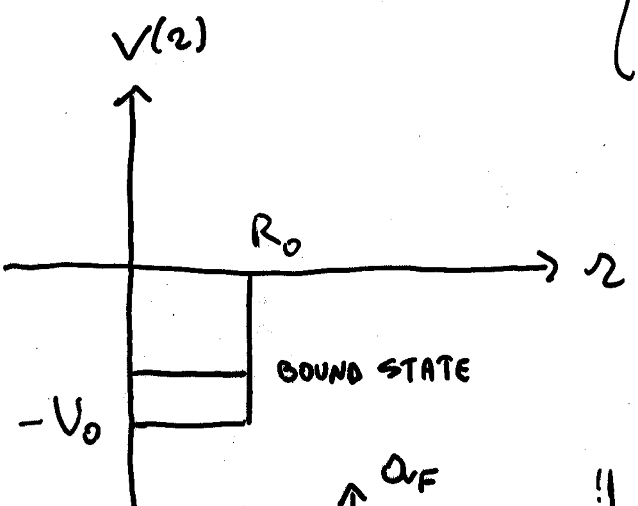
BCS - BEC CROSSOVER
WITH FERMI GASES

LET US CONSIDER A GAS OF FERMIONS WITH __2 SPIN COMPONENTS__

$$\sigma = \uparrow, \downarrow \qquad N = N_\uparrow + N_\downarrow$$

AND THE FOLLOWING INTERPARTICLE POTENTIAL

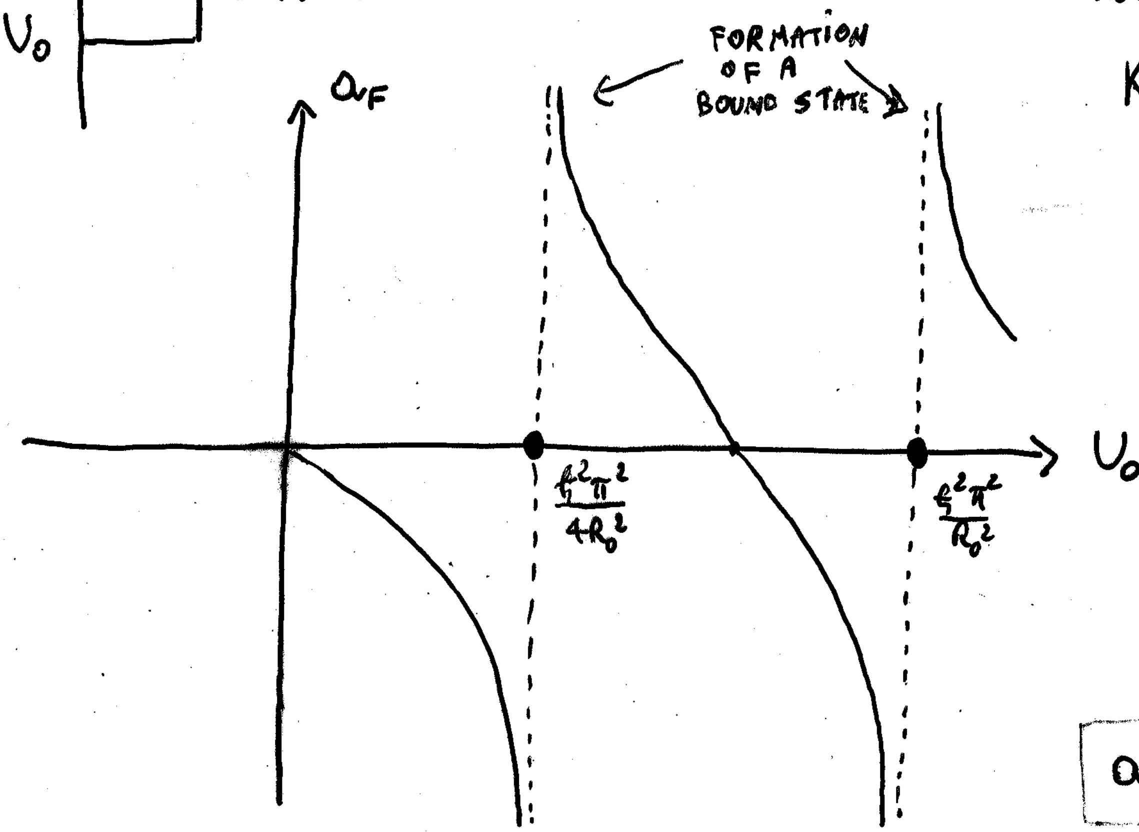
$$V(r) = \begin{cases} -U_0 & \text{FOR } 0 \leq r \leq R_0 \\ 0 & \text{FOR } r > R_0 \end{cases}$$



THE S-WAVE SCATTERING LENGTH IS GIVEN BY

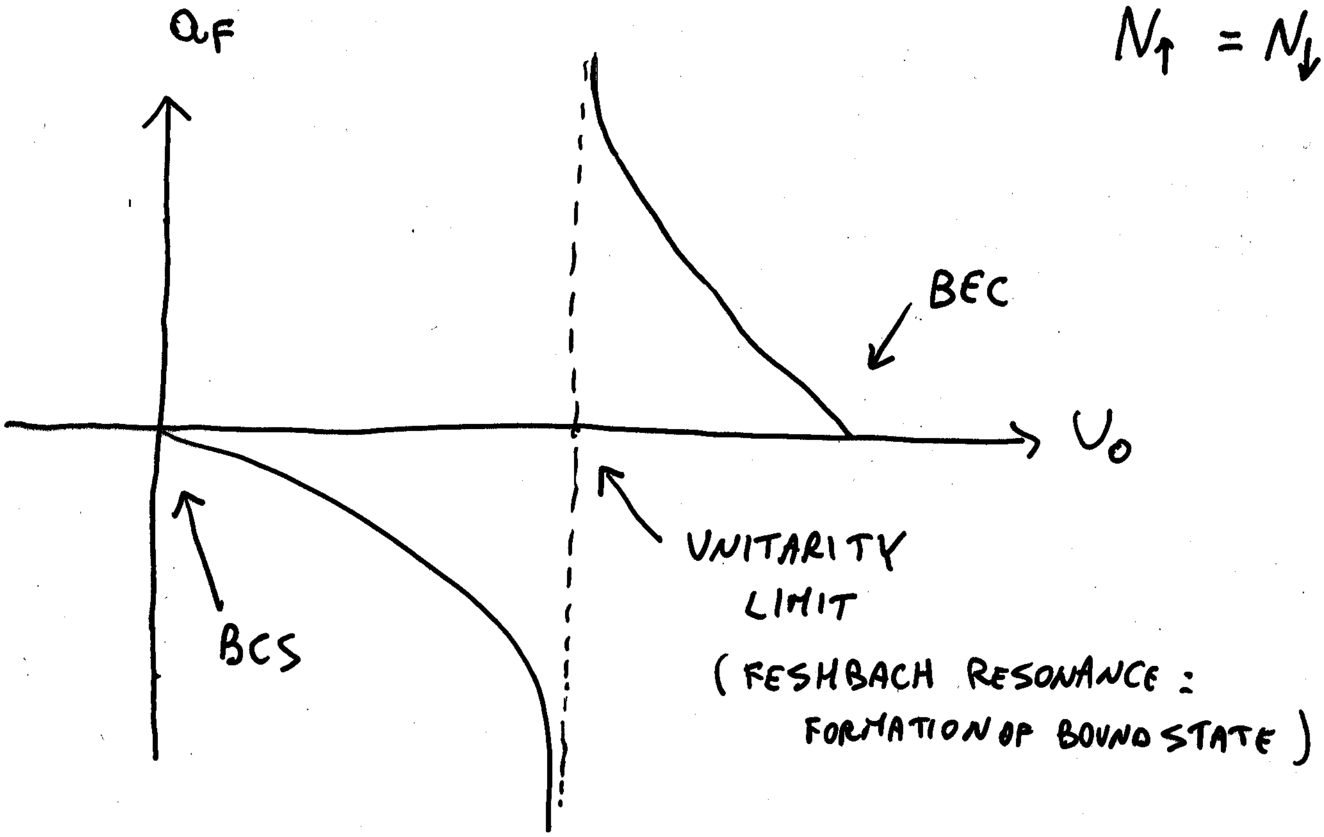
$$a_F = R_0 \left[1 - \frac{tg(K_0 R_0)}{K_0 R_0} \right]$$

$$K_0 = \sqrt{\frac{m U_0}{\hbar^2}}$$



$$\boxed{a_F = a_{\uparrow\downarrow}}$$

$N_\uparrow = N_\downarrow$



BCS REGIME : FERMI GAS WITH A WEAK ATTRACTIVE INTER-ATOMIC STRENGTH

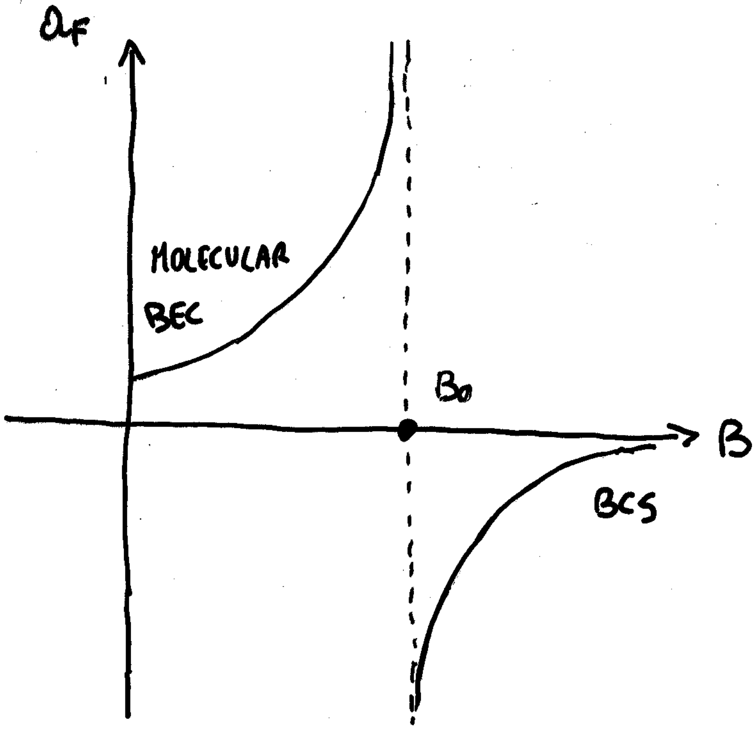
(COOPER PAIRS)

UNITARITY LIMIT : $a_F \to \pm \infty$

BEC REGIME : GAS OF BOSE CONDENSED MOLECULES (DIMERS) WITH SPIN INTEGER.

REMARK : INTERACTION AT SHORT DISTANCE IS ACTIVE ONLY IN PRESENCE OF TWO SPIN SPECIES (CONSEQUENCE OF PAULI PRINCIPLE).

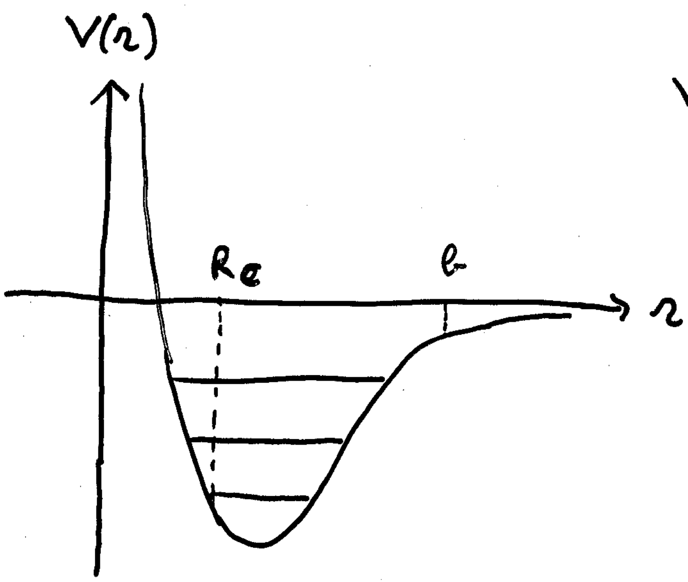
FESHBACH RESONANCE WITH AN EXTERNAL MAGNETIC FIELD



$B_0 \simeq 222$ GAUSS $\quad {}^{40}K$

$B_0 \simeq 834$ GAUSS $\quad {}^{6}Li$

B = EXTERNAL MAGNETIC FIELD WHICH MODIFIES THE INTER-ATOMIC POTENTIAL $V(r)$



$$V(r) = \frac{C_{12}}{r^{12}} - \frac{C_6}{r^6} \simeq \begin{cases} \infty & 0 \le r \le R_e \\ -\frac{C_6}{r^6} & r > R_e \end{cases}$$

$$b = \left(\frac{2m\,C_6}{\hbar^2}\right)^{1/4} \simeq 1-10 \text{ nm}$$

$$a_F \simeq \frac{1}{2} b \left[1 - tg\left(\frac{b^2}{2R_e} - \frac{3\pi}{8}\right)\right]$$

$$\vec{F} = \vec{S} + \vec{I} \qquad |F, m_F\rangle$$

$\boxed{{}^{6}Li}$

$\vec{I} = 1, \quad \vec{S} = \frac{1}{2}, \quad \vec{F} = \frac{1}{2}, \frac{3}{2}$

$m_F = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

$\boxed{{}^{40}K}$

$\vec{I} = 4, \quad \vec{S} = \frac{1}{2}, \quad \vec{F} = \frac{7}{2}, \frac{9}{2}$

$m_F = -\frac{9}{2}, -\frac{7}{2}, \dots, \frac{7}{2}, \frac{9}{2}$

FORMULAS FOR WEAKLY-INTERACTING GAS AT ZERO TEMPERATURE

- FOR WEAKLY-INTERACTING **FERMIONS** :

$a_F = a_{\uparrow\downarrow}$

$$\mathcal{E} = \frac{E}{N} = \frac{3}{5}\,\varepsilon_F \left(1 + \frac{10}{9\pi} k_F a_F + \dots \right)$$

HUANG-YANG (1957)

WHERE

$$\begin{cases} \varepsilon_F = \dfrac{\hbar^2}{2m} \left(3\pi^2 n\right)^{2/3} & \text{FERMI ENERGY} \\ k_F = \left(3\pi^2 n\right)^{1/3} & \text{FERMI WAVE NUMBER} \end{cases}$$

- FOR WEAKLY-INTERACTING **BOSONS** :

$$\mathcal{E} = \frac{E}{N} = \frac{4\pi\hbar^2 a_M}{m_M}\, n_M \left(1 + \frac{32}{3\sqrt{\pi}} \left(a_M n_M^{1/3}\right)^{3/2} + \dots \right)$$

LEE-YANG-HUANG (1957)

WHERE

$$\begin{cases} m_M = 2m & \text{MOLECULAR MASS} \\ n_M = \dfrac{n}{2} = \dfrac{n_\uparrow + n_\downarrow}{2} = \dfrac{2 n_\uparrow}{2} = n_\uparrow = n_\downarrow & \text{MOLECULAR DENSITY} \end{cases}$$

$N_\uparrow = N_\downarrow$

- WHAT ABOUT a_M ?

EBCS MEAN FIELD THEORY : $a_M = 2\, a_F$

4-BODY CALCULATIONS : $\boxed{a_M = 0.6\, a_F}$

MONTE CARLO CALCULATIONS CONFIRM 4-BODY THEORY !

BEHAVIOR AT RESONANCE (UNITARITY)

AT RESONANCE $\quad k_F |a_F| \gg 1$

THE SYSTEM IS <u>STRONGLY CORRELATED</u> BUT ITS PROPERTIES DO NOT DEPEND ON a_F:

UNIVERSALITY!

IN GENERAL

$$\boxed{\varepsilon = \frac{E}{N} = \frac{\hbar^2}{2m} F(n, a_F)}$$

WHERE $F(n, a_F)$ IS AN UNKNOW FUNCTION OF DENSITY n AND SCATTERING LENGTH a_F

$$\lim_{a_F \to \infty} F(n, a_F) = C \, n^{2/3}$$

UNKNOWN CONSTANT (ADIMENSIONAL)

ONLY IN THIS WAY $\varepsilon = \frac{E}{N}$ HAS THE CORRECT DIMENSIONS

USUALLY ONE SETS:

$$C = \frac{3}{5} (3\pi^2)^{2/3} \, \xi$$

WHERE:

EBCS MEAN FIELD THEORY: $\quad \xi = 0.64$

MONTE CARLO RESULTS: $\quad \boxed{\xi = 0.42}$

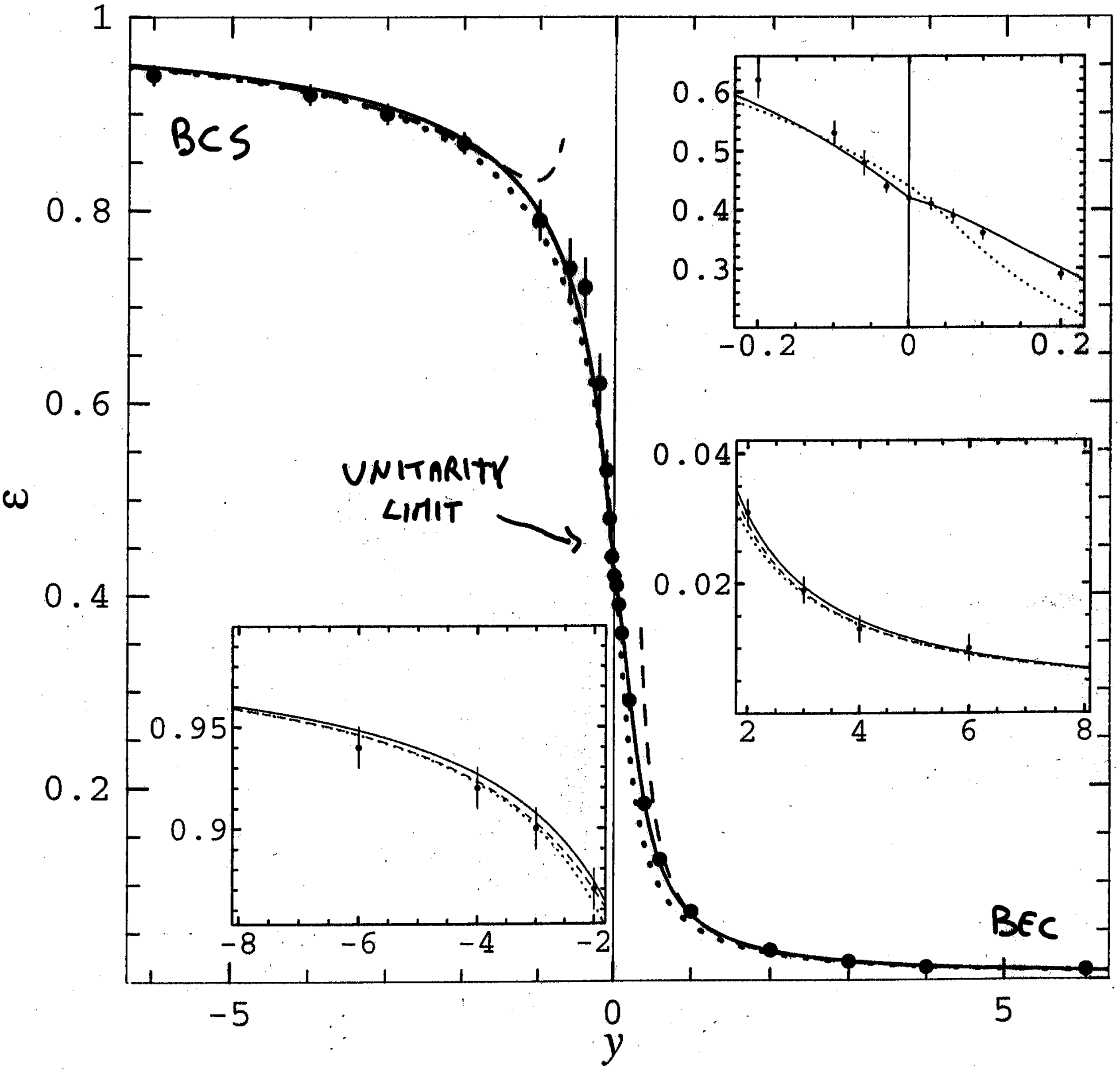
BCS - BEC WITH MONTE CARLO

$$\varepsilon = \frac{\text{ENERGY PER PARTICLE}}{\text{FERMI ENERGY}} \cdot \frac{5}{3}$$

$$\boxed{y = \frac{1}{K_F a_F}}$$

INVERSE INTERACTION PARAMETER

$$K_F = (3\pi^2 n)^{1/3}$$



• MONTE CARLO DATA OF G.E. ASTRAKHARCHIK ET. AL
PRL <u>93</u>, 200404 (2004).

—— INTERPOLATING FORMULA OF MANINI AND SALASNICH
PRA <u>71</u>, 033625 (2005).

EQUATION OF STATE IN THE BCS-BEC CROSSOVER AT ZERO TEMPERATURE

2 COMPONENTS FERMI GAS WITH $n = n_\uparrow + n_\downarrow = 2n_\uparrow$

LOCAL DENSITY $n_\uparrow = n_\downarrow$

a_F S-WAVE ATOM-ATOM SCATTERING LENGTH

AT ZERO TEMPERATURE $(T=0)$:

BULK ENERGY PER PARTICLE

$$\boxed{\mathcal{E} = \frac{3}{5}\,\varepsilon_F\, f(y)}$$

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}$$ FERMI ENERGY

$$k_F = \left(3\pi^2 n\right)^{1/3}$$ FERMI WAVE NUMBER

$$y = \frac{1}{k_F a_F} = \frac{1}{\left(3\pi^2 n\right)^{1/3} a_F}$$ INVERSE INTERACTION PARAMETER

ASYMPTOTIC FORMULAS:

$$f(y) = \begin{cases} 1 + \dfrac{10}{9\pi}\dfrac{1}{y} + \cdots & y \ll -1 \quad \text{BCS} \\ \xi & y = 0 \quad \text{UNITARITY} \\ \dfrac{5}{18\pi}\dfrac{a_M}{a_F}\dfrac{1}{y}\left(1 + \dfrac{128}{15\sqrt{6\pi^2}}\left(\dfrac{a_M}{a_F}\right)^{3/2}\dfrac{1}{y^{3/2}} + \cdots\right) & y \gg 1 \quad \text{BEC} \end{cases}$$

UNIVERSAL FUNCTION

FROM FIXED-NODE DIFFUSION MONTE-CARLO CALCULATIONS

$$\xi = 0.42 \pm 0.01$$

G.E. ASTRAKHARCHIK ET AL., PRL <u>93</u> 200404 (2004)

FROM ASYMPTOTIC FORMULAS AND MC DATA
ONE GETS A FITTING FORMULA : *

$$f(y) = \alpha_1 - \alpha_2 \arctg\left[\alpha_3 \, y \, \frac{\beta_1 + |y|}{\beta_2 + |y|}\right]$$

$\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ INTERPOLATING PARAMETERS.

THEN MANY ZERO-TEMPERATURE THERMODYNAMICS EXPRESSIONS CAN BE FOUND :

$$\mu = \frac{\partial(n\,\mathcal{E})}{\partial n} = \varepsilon_F \left(f(y) - \frac{y}{5} f'(y) \right)$$

CHEMICAL POTENTIAL

$$P = n^2 \frac{\partial \mathcal{E}}{\partial n} = n\,\varepsilon_F \left(\frac{2}{5} f(y) - \frac{y}{5} f'(y) \right)$$

PRESSURE

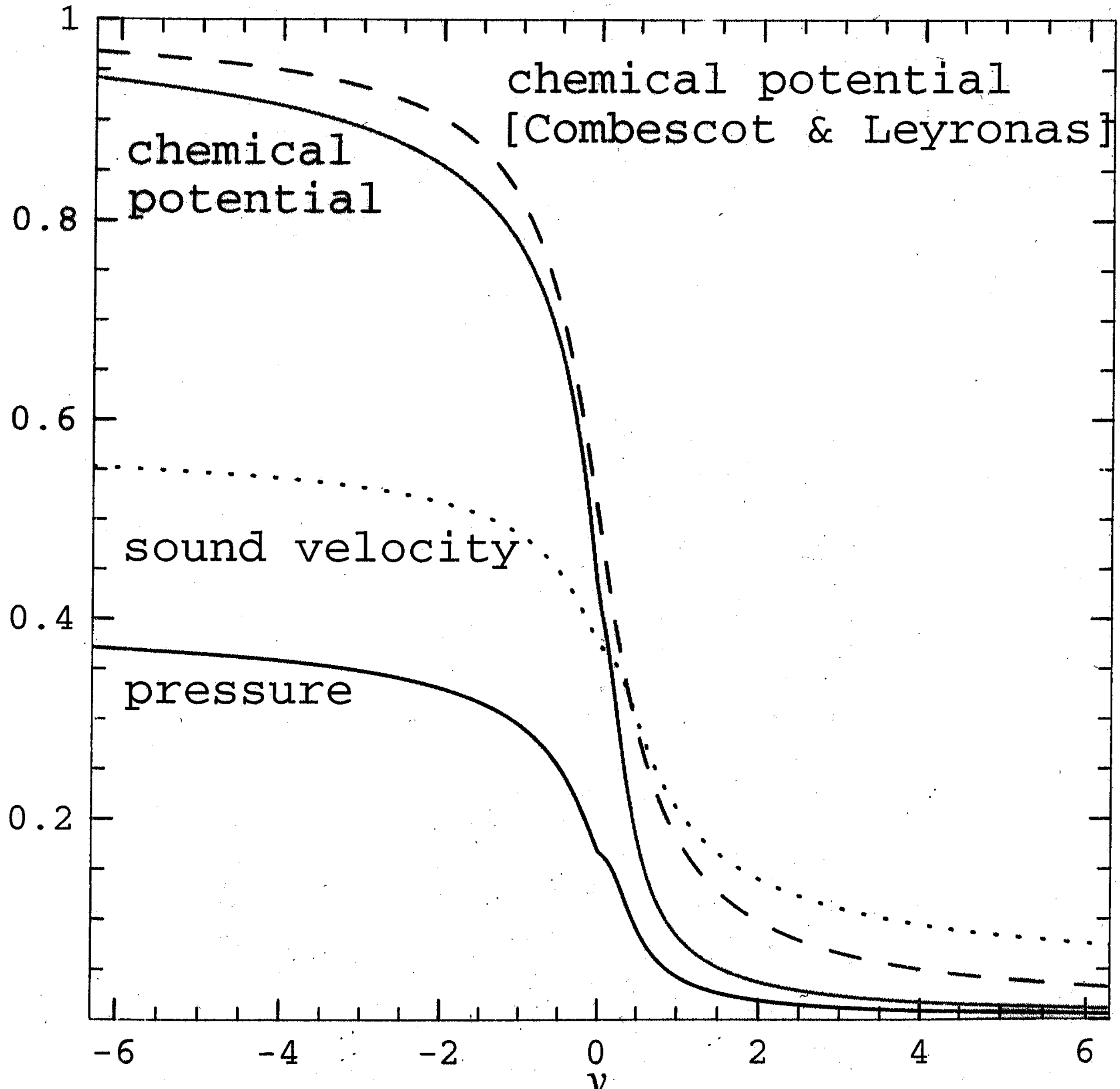
$$c_s^2 = \frac{n}{m} \frac{\partial \mu}{\partial n} = \frac{1}{3} f(y) - \frac{1}{5} y f'(y) + \frac{1}{30} y^2 f''(y)$$

SOUND VELOCITY

FOR THE UNIFORM FERMI SYSTEM
(BULK SYSTEM).

* N. MANINI AND L. SALASNICH, PRA 71, 033625 (2005).

THERMODYNAMIC QUANTITIES IN THE BCS - BEC CROSSOVER



BCS UNITARITY BEC

$$y = \frac{1}{K_F a_F}$$

$K = (3\pi^2 n)^{1/3}$ FERMI WAVE NUMBER

a_F FERMI-FERMI SCATTERING LENGTH

BCS-BEC CROSSOVER AND HYDRODYNAMIC EQUATIONS

THE HYDRODYNAMIC EQUATIONS OF <u>SUPERFLUIDS</u> ARE

$(T=0)$

$$\begin{cases} \frac{\partial}{\partial t} n + \vec{\nabla} \cdot (n \vec{v}) = 0 \\ m \frac{\partial}{\partial t} \vec{v} + \vec{\nabla} \left[U(\vec{r}) + \mu(n) + \frac{1}{2} m v^2 \right] = \vec{0} \end{cases}$$

EULER EQUATIONS FOR A NOT-VISCOUS AND IRROTATIONAL FLUID.

THEY DESCRIBE COLLECTIVE PROPERTIES OF <u>BOTH</u> <u>FERMI</u> AND <u>BOSE</u> SUPERFLUIDS.

<u>MAIN IDEA</u> : $\vec{v}$ IRROTATIONAL (i.e. $\vec{\nabla} \wedge \vec{v} = \vec{0}$) IMPLIES

$$\vec{v} = \vec{\nabla} S$$

AND S IS THE PHASE OF THE CONDENSATE WAVE FUNCTION.

FOR <u>FERMIONS</u>

$$\vec{v} = \frac{\hbar}{2m} \vec{\nabla} \theta$$

NOTE THIS !

WITH

$$\langle \hat{\psi}_{\downarrow}(\vec{r},t)\, \hat{\psi}_{\uparrow}(\vec{r},t) \rangle = \Xi(\vec{r},t) = |\Xi(\vec{r},t)|\, e^{i\theta(\vec{r},t)}$$

CONDENSATE WAVE FUNCTION OF PAIRS

$\hat{\psi}_{\uparrow}(\vec{r},t)$ FERMIONIC FIELD OPERATOR

<u>KEY INGREDIENT</u> : THE BULK CHEMICAL POTENTIAL $\boxed{\mu(n)}$

$$\mu = \mu(n; a_F) .$$

TD GLE AT T=0

THE TIME-DEPENDENT GINZBURG-LANDAU EQUATION AT ZERO TEMPERATURE :

$$i\hbar \frac{\partial}{\partial t} \Xi = \left[-\frac{\hbar^2}{4m} \nabla^2 + 2\,U(\vec{r}) + 2\,\mu(n) \right] \Xi$$

WHERE :

$$\Xi(\vec{r},t) = \sqrt{\frac{n(\vec{r},t)}{2}}\, e^{i\theta(\vec{r},t)}$$

GINZBURG-LANDAU ORDER PARAMETER

AND THE POSITION

$$\vec{v} = \frac{\hbar}{2m} \vec{\nabla}\theta$$

GIVES THE HYDRODYNAMIC EQUATIONS OF SUPERFLUIDS (T=0)

WITH THE ADDITIONAL QUANTUM-PRESSURE TERM

$$-\frac{\hbar^2}{8m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} .$$

NOTE THAT

$$\int |\Xi(\vec{r},t)|^2\, d^3\vec{r} = \frac{N}{2}$$

TOTAL NUMBER OF PAIRS

WHILE

$$\int |\Xi(\vec{r},t)|^2\, d^3\vec{r} = \frac{N_0}{2}$$

NUMBER OF PAIRS IN THE CONDENSATE

REMARK: THE TD GLE IS VALID IF

$$|\Xi(\vec{r},t)| \ll |\Xi(\vec{0},0)|$$

i.E. FOR COLLECTIVE PHENOMENA WITH LONG-WAVE-LENGTH LOW-FREQUENCY EXCITATIONS.