# Dimensional reduction and localization of a Bose-Einstein condensate in a quasi-1D bichromatic optical lattice

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# Summary

- 1. 1D GPE with a quasi-periodic bichromatic potential
- 2. Derivation of the 1D GPE
- 3. Numerical Results
- Conclusions

# 1D Gross-Pitaevskii equation with a quasi-periodic bichromatic potential (I)

In 1958 Anderson[1] predicted the **localization** of the electronic wave function in a disordered potential.
In the last twenty years the phenomenon of **localization due to disorder** was experimentally observed in electromagnetic waves[2], in sound waves[3], and also in quantum matter waves[4].
In 2008 at INO-CNR of Florence[5] it was observed the localization of a non-interacting Bose-Einstein condensate (BEC) of $^{39}$K atoms in a 1D potential created by two optical-lattice potentials with different amplitudes and wavelengths.

[1] P.W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] D. S. Wiersma *et al.*, Nature **390**, 671 (1997)
[3] R. L. Weaver *et al.*, Wave Motion **12**, 129 (1990).
[4] J. Billy *et al.*, Nature **453**, 891 (2008).
[5] G. Roati *et al.*, Nature **453**, 895 (2008).

# 1D Gross-Pitaevskii equation with a quasi-periodic bichromatic potential (II)

Following the details of the experiment in Florence, we model the dynamics of a trapped BEC of $N$ atoms by using the following adimensional 1D Gross-Pitaevskii equation (1D GPE)

$$i\frac{\partial}{\partial t}\phi(z,t) = \left[ -\frac{1}{2}\partial_z^2 + V(z) + g|\phi(z,t)|^2 \right]\phi(z,t) \ , \qquad (1)$$

where

$$V(z) = \sum_{i=1}^{2} \frac{4\pi^2 s_i}{\lambda_i^2} \cos^2\left(\frac{2\pi}{\lambda_i} z\right), \qquad (2)$$

is the quasi-periodic bichromatic axial potential, with $\phi(z,t)$ the axial wave function of the Bose condensate normalized to one, i.e.

$$\int_{-\infty}^{\infty} dz \ |\phi(z,t)|^2 = 1 \ . \qquad (3)$$

Here $g = 2Na_s/a_\perp$ is the dimensionless interaction strength with $a_s$ the inter-atomic scattering length and $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$ the characteristic harmonic length of the transverse harmonic confinement.

# 1D Gross-Pitaevskii equation with a quasi-periodic bichromatic potential (III)

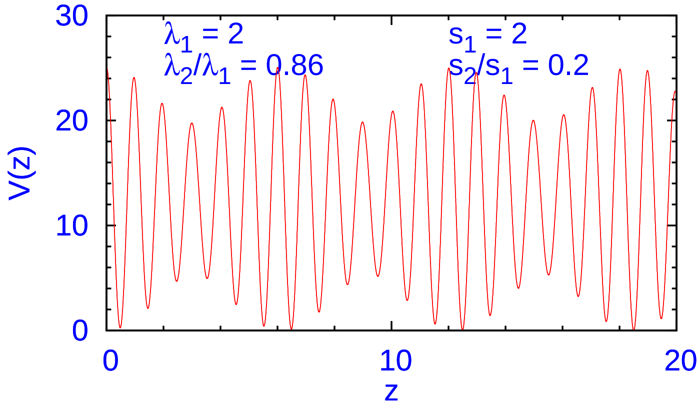
Notice that with a single periodic potential the linear Schrödinger equation permits only delocalized states in the form of Bloch waves. Localization is possible in the linear Schrödinger equation due to the "disorder" introduced through a second periodic component:

$$V(z) = \frac{4\pi^2 s_1}{\lambda_1^2}\cos^2\left(\frac{2\pi}{\lambda_1}z\right) + \frac{4\pi^2 s_2}{\lambda_2^2}\cos^2\left(\frac{2\pi}{\lambda_2}z\right) \tag{4}$$

The optical potential of wavelength $\lambda_1$ is used to create a primary lattice that is weakly perturbed by a secondary lattice of wavelength $\lambda_2$. Moreover, to obtain "quasi-disorder" the ratio $\lambda_2/\lambda_1$ should not be commensurable. In practice we use $\lambda_2/\lambda_1 = 0.86$ that is close to the experimental value $\lambda_2/\lambda_1 = 0.835$ of Roati *et al.*[6]

[6] G. Roati *et al.*, Nature **453**, 895 (2008).

# 1D Gross-Pitaevskii equation with a quasi-periodic bichromatic potential (IV)




Bichromatic optical axial potential. All variables are expressed in dimensionless units. Adapted from **S.K. Adhikari and LS, Phys. Rev. A 80, 023606 (2009)**.

# Derivation of the 1D GPE (I)

Before solving the 1D GPE, let us analyze its derivation from the quantum field theory of many-body systems.
The quantum many-body Hamiltonian of interacting identical bosons is given by

$$\hat{H} = \int d^3\mathbf{r}\ \hat{\psi}^+(\mathbf{r}) \left[ -\frac{1}{2}\nabla^2 + U(\mathbf{r}) \right] \hat{\psi}(\mathbf{r}) + \int d^3\mathbf{r}\ d^3\mathbf{r}'\ \hat{\psi}^+(\mathbf{r})\hat{\psi}^+(\mathbf{r}')W(\mathbf{r},\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r}) \quad (5)$$

where $\hat{\psi}(\mathbf{r},t)$ is the bosonic field operator.
In our case the external trapping potential reads

$$U(\mathbf{r}) = \frac{1}{2}(x^2 + y^2) + V(z) \ , \quad (6)$$

corresponding to a harmonic transverse confinement of frequency $\omega_\perp$ with characteristic length $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$ and the axial optical lattice $V(z)$ of Eq. (2).

# Derivation of the 1D GPE (II)

In addition, due to the fact that the system is made of dilute and ultracold atoms, we consider a contact interaction between bosons, i.e.

$$W(\mathbf{r}-\mathbf{r}') = \gamma\,\delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{7}$$

with $\delta^{(3)}(\mathbf{r})$ the Dirac delta function and

$$\gamma = 2\frac{a_s}{a_\perp} \tag{8}$$

the adimensional strength of the boson-boson interaction, proportional to the s-wave scattering length $a_s$ of the inter-atomic potential $W(\mathbf{r},\mathbf{r}')$. Taking into account Eqs. (6) and (7), the Heisenberg equation of motion of the field operator

$$i\frac{\partial}{\partial t}\hat{\psi} = [\hat{\psi},\hat{H}] \tag{9}$$

gives

$$i\frac{\partial}{\partial t}\hat{\psi}(\mathbf{r},t) = \left[-\frac{1}{2}\nabla^2 + \frac{1}{2}\left(x^2+y^2\right) + V(z) + 2\pi\gamma\hat{\psi}^{+}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\right]\hat{\psi}(\mathbf{r},t)\,. \tag{10}$$

# Derivation of the 1D GPE (III)

In the superfluid regime, where the many-body quantum state $|QS\rangle$ of the system can be approximated by a coherent state[7] $|CS\rangle$, i.e. such that

$$\hat{\psi}(\mathbf{r},t)|CS\rangle = \psi(\mathbf{r},t)|CS\rangle \ , \tag{11}$$

the Heisenberg equation of motion (10) becomes the familiar 3D Gross-Pitaevskii equation (3D GPE)[8]

$$i\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[-\frac{1}{2}\nabla^2 + \frac{1}{2}\left(x^2+y^2\right) + V(z) + 2\pi\gamma|\psi(\mathbf{r},t)|^2\right]\psi(\mathbf{r},t) \ , \tag{12}$$

where $\psi(\mathbf{r},t)$ is a complex wavefunction normalized to the total number $N$ of bosons, i.e.

$$\int d^3\mathbf{r} \ |\psi(\mathbf{r},t)|^2 = N \ . \tag{13}$$

[7] LS, Quantum Physics of Light and Matter. A Modern Introduction to Photons, Atoms and Many-Body Systems (Springer, 2014).

[8] V.P. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Oxford Univ. Press, Oxford, 2003).

# Derivation of the 1D GPE (IV)

The time-dependent 3D GPE (12) is the Euler-Lagrange equation of the action functional

$$S = \int dt\, d^3\mathbf{r}\ \mathcal{L} \qquad (14)$$

with Lagrangian density

$$\mathcal{L} = \psi^* \left[ i\frac{\partial}{\partial t} + \frac{1}{2}\nabla^2 \right] \psi - \frac{1}{2}\left(x^2+y^2\right)|\psi|^2 - V(z)|\psi|^2 - \pi\gamma|\psi|^4\,. \qquad (15)$$

To perform the **dimensional reduction** we suppose that

$$\psi(\mathbf{r},t) = \frac{N^{1/2}}{\pi^{1/2}\sigma(z,t)} \exp\left[ -\left( \frac{x^2+y^2}{2\sigma(z,t)^2} \right) \right] \phi(z,t)\ , \qquad (16)$$

where $\sigma(z,t)$ and $\phi(z,t)$ account respectively for the transverse width and for the axial bosonic wavefunction. We apply this Gaussian variational ansatz to the action functional of the 3D GPE.

# Derivation of the 1D GPE (V)

Integrating over $x$ and $y$ and neglecting the derivatives of $\sigma(z,t)$, we obtain the effective 1D action[9]

$$S_e = \int dt\, dz\ \mathcal{L}_e \tag{17}$$

with the effective 1D Lagrangian density

$$\mathcal{L}_e = \phi^* \left[ i\frac{\partial}{\partial t} + \frac{1}{2}\partial_z^2 \right] \phi - V(z)|\phi|^2 - \frac{1}{2}\left( \frac{1}{\sigma^2} + \sigma^2 \right) |\phi|^2 + \frac{g}{2\sigma^2}|\phi|^4 \ , \tag{18}$$

and

$$g = N\gamma = \frac{2Na_s}{a_\perp} \ . \tag{19}$$

[9] LS, Laser Phys. **12**, 198 (2002); LS, A. Parola, and L. Reatto, Phys. Rev. A **65**, 043614 (2002).

# Derivation of the 1D GPE (VI)

Calculating the Euler-Lagrange equations of both $\phi(z,t)$ and $\sigma(z,t)$ one finds[10]

$$i\frac{\partial}{\partial t}\phi = \Big[-\frac{1}{2}\partial_z^2 + V(z) + \frac{1}{2}\left(\frac{1}{\sigma^2}+\sigma^2\right) + \frac{g}{\sigma^2}\,|\phi|^2\Big]\phi\ , \qquad (20)$$

and

$$\sigma^4 = 1 + g|\phi|^2 + (\partial_z\sigma)^2 + \frac{\sigma^3}{|\phi|^2}\partial_z\left(\frac{\partial_z\sigma}{\sigma^2}|\phi|^2\right)\ . \qquad (21)$$

Neglecting the spatial derivatives of $\sigma(z,t)$ (adiabatic approximation) the last equation becomes

$$\sigma = \left(1 + g|\phi|^2\right)^{1/4}\ . \qquad (22)$$

[10] LS, A. Parola, and L. Reatto, Phys. Rev. A **65**, 043614 (2002); LS, B.A. Malomed, and F. Toigo, Phys. Rev. A **76**, 063614 (2007).

# Derivation of the 1D GPE (VII)

Eqs. (20) and (22) give the 1D nonpolynomial Schrödinger equation (NPSE)[11]

$$i\frac{\partial}{\partial t}\phi = \Big[-\frac{1}{2}\partial_z^2 + V(z) + \frac{1}{2}\left(\frac{1}{\sqrt{1+g|\phi|^2}} + \sqrt{1+g|\phi|^2}\right) + \frac{g|\phi|^2}{\sqrt{1+g|\phi|^2}}\Big]\phi \tag{23}$$

Finally, only under the condition

$$g|\phi(z,t)|^2 \ll 1 \tag{24}$$

one finds $\sigma = 1$ (i.e. $\sigma = a_\perp$ is dimensional units) and Eq. (20) becomes the 1D Gross-Pitaevskii equation (1D GPE)

$$i\frac{\partial}{\partial t}\phi = \Big[-\frac{1}{2}\partial_z^2 + V(z) + 1 + g\,|\phi|^2\Big]\phi\,. \tag{25}$$

Notice that NPSE, Eq. (23), has been used by many authors to study quasi-1D BECs with a transverse width $\sigma \neq 1$.

[11] LS, Laser Phys. **12**, 198 (2002); LS, A. Parola, and L. Reatto, Phys. Rev. A **65**, 043614 (2002).

# Numerical results (I)

We perform the **numerical simulation** of the time-dependent 1D GPE

$$i\frac{\partial}{\partial t}\phi(z,t) = \left[ -\frac{1}{2}\partial_z^2 + V(z) + g|\phi(z,t)|^2 \right]\phi(z,t) \ , \qquad (26)$$

where

$$V(z) = \frac{4\pi^2 s_1}{\lambda_1^2}\cos^2\left(\frac{2\pi}{\lambda_1}z\right) + \frac{4\pi^2 s_2}{\lambda_2^2}\cos^2\left(\frac{2\pi}{\lambda_2}z\right) \ , \qquad (27)$$

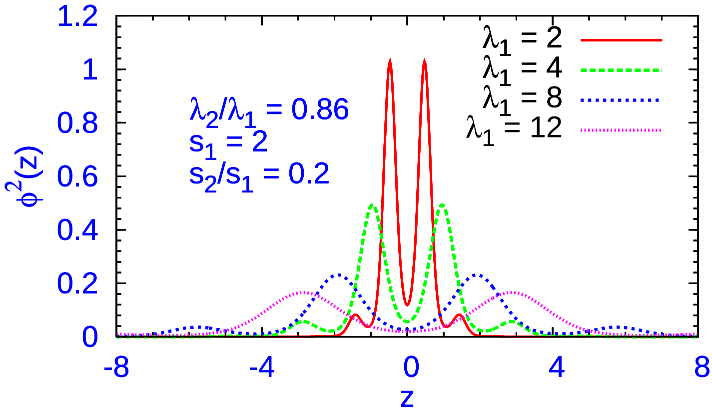
employing real-time propagation with finite-difference Crank-Nicholson method[12]. We choose the initial condition

$$\phi(z,0) = \frac{1}{\pi^{1/4}\eta^{1/2}} \ e^{-z^2/(2\eta^2)} \ , \qquad (28)$$

with $\eta = 1$ and imposing $\phi(\pm z_B, t) = 0$ with $z_B = 100$. We stop the dynamics when a "stationary" configuration is reached.

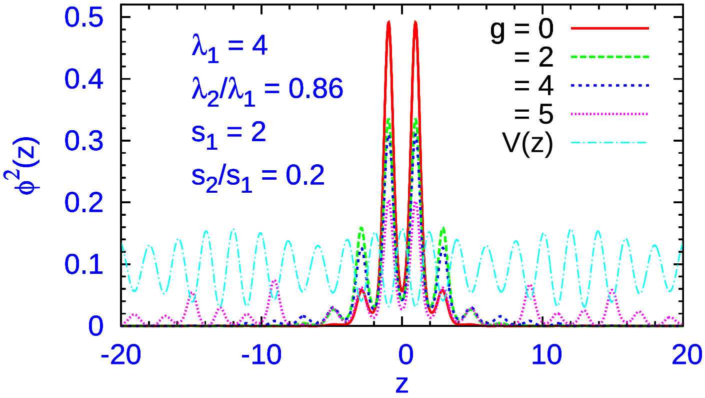
[12] P. Muruganandam and S.K. Adhikari, Computer Phys. Commun. **180**, 1888-1912 (2009).

# Numerical results (II)



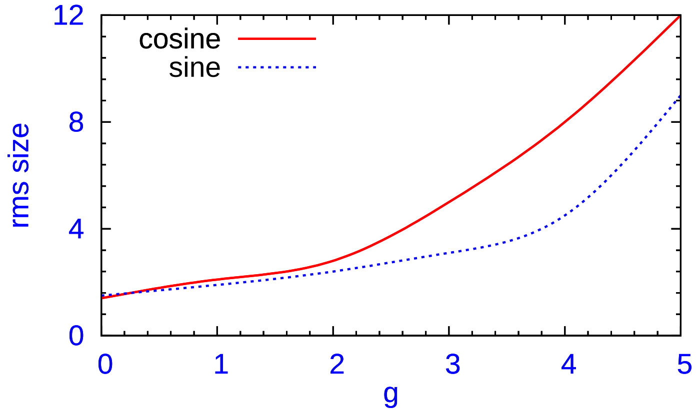

Typical density distribution $\phi^2(z)$ vs. $z$ for the non interacting BEC. The curves correspond to different values of the wavelength $\lambda_1$ of the main optical potential, but the ratio $\lambda_2/\lambda_1$ is kept constant. Adapted from **S.K. Adhikari and LS, Phys. Rev. A 80, 023606 (2009)**.

# Numerical results (III)




Typical density distribution $\phi^2(z)$ vs. $z$ for the interacting BEC with different values of the interaction strength $g = 2Na_s/a_\perp$. The quasi-periodic optical-lattice potential $V(z)$ is plotted in arbitrary units with $\lambda_1 = 4, \lambda_2/\lambda_1 = 0.86, s_1 = 2, s_2/s_1 = 0.2$. Adapted from **S.K. Adhikari and LS, Phys. Rev. A 80, 023606 (2009)**.

# Numerical results (IV)




The root-mean-square (rms) size vs. interaction strength $g$ of the BEC in the quasi-periodic potential (2) (cosine), and a similar potential where cosines are substituted by sines (sine), with $\lambda_1 = 4, \lambda_2/\lambda_1 = 0.86, s_1 = 2, s_2/s_1 = 0.2$. Adapted from **S.K. Adhikari and LS, Phys. Rev. A 80, 023606 (2009)**.

# Conclusions

- The 1D GPE can be derived from the 3D GPE through a 1D nonpolynomial Schrödinger equation (NPSE) under strict conditions.
- 1D GPE with a quasi-periodic bichromatic external potential $V(z)$ has been solved numerically.
- In the absence of nonlinearity, the time-dependent 1D GPE (i.e. the time-dependent 1D LSE) with $V(z)$ gives **localized "stationary" solutions**.
- With a finite weak nonlinearity of strength $g$, the time-dependent 1D GPE with $V(z)$ still gives **localized "stationary" solutions**.
- When $g > 5$ our numerical solutions of the time-dependent 1D GPE with $V(z)$ suggest that the localization is precticaly lost.