# Quantum fluctuations in two-dimensional Fermi superfluids

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Trieste, February 17, 2016

Work done in collaboration with
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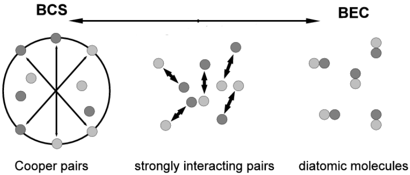
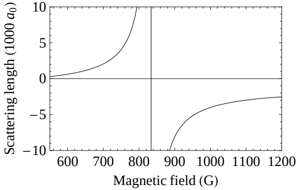
# Summary

- BCS-BEC crossover in 3D and 2D
- Quantum fluctuations in $D$-dimensions
- New results for 2D BCS-BEC crossover
- Conclusions

# BCS-BEC crossover in 3D and 2D (I)

In 2004 the 3D BCS-BEC crossover has been observed with **ultracold gases made of two-component fermionic $^{40}$K or $^{6}$Li alkali-metal atoms.**[1]




This crossover is obtained by using a Fano-Feshbach resonance to change the 3D s-wave scattering length $a_s$ of the inter-atomic potential

$$a_s = a_{bg}\left(1 + \frac{\Delta_B}{B - B_0}\right) , \tag{1}$$

where $B$ is the external magnetic field.

[1] C.A. Regal et al., PRL **92**, 040403 (2004); M.W. Zwierlein et al., PRL **92**, 120403 (2004); J. Kinast et al., PRL **92**, 150402 (2004).

# BCS-BEC crossover in 3D and 2D (II)

The 3D crossover from a BCS superfluid ($a_s < 0$) to a BEC of molecular pairs ($a_s > 0$) has been investigated experimentally around a Fano-Feshbach resonance, where the 3D s-wave scattering length $a_s$ diverges, and it has been shown that the system is (meta)stable.
The detection of quantized vortices under rotation[2] has clarified that this dilute gas of ultracold atoms is superfluid.
Usually the 3D BCS-BEC crossover is analyzed in terms of

$$y = \frac{1}{k_F a_s} \tag{2}$$

the inverse scaled interaction strength, where $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wave number and $n$ the total density.
The system is dilute because $r_e k_F \ll 1$, with $r_e$ the effective range of the inter-atomic potential.

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[2] M. W. Zwierlein *et al.*, Nature **435**, 1047 (2005); M.W. Zwierlein *et al.*, Science **311**, 492 (2006).

# BCS-BEC crossover in 3D and 2D (III)

In the last few years also the 2D BEC-BEC crossover has been achieved[3] with a **quasi-2D Fermi gas of two-component $^6$Li atoms**. In the 2D attractive system fermions always form biatomic molecules with bound-state energy

$$\epsilon_B \simeq \frac{\hbar^2}{m {a_s}^2} \, , \tag{3}$$

where $a_s$ is the 2D s-wave scattering length.
Both in 3D and 2D the fermionic single-particle spectrum is given by

$$E_{sp}(k) = \sqrt{\left( \frac{\hbar^2 k^2}{2m} - \mu \right)^2 + \Delta^2} \, , \tag{4}$$

where $\Delta$ is the energy gap and $\mu$ is the chemical potential: $\mu > 0$ corresponds to the BCS regime while $\mu < 0$ corresponds to the BEC regime. Moroever, in the deep BEC regime $\mu \to -\epsilon_B/2$.

[3] V. Makhalov et al. PRL **112**, 045301 (2014); M.G. Ries et al., PRL **114**, 230401 (2015); K. Fenech et al., PRL **116**, 045302 (2016); I. Boettcher et al., PRL **116**, 045303 (2016).

# Quantum fluctuations in $D$-dimensions (I)

We adopt the formalism of functional integration[4]. The partition function $\mathcal{Z}$ of the uniform system with fermionic fields $\psi_s(\mathbf{r},\tau)$ at temperature $T$, in a $D$-dimensional volume $L^D$, and with chemical potential $\mu$ reads

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \ \exp\left\{ -\frac{S}{\hbar} \right\} , \tag{5}$$

where ($\beta \equiv 1/(k_B T)$ with $k_B$ Boltzmann's constant)

$$S = \int_0^{\hbar\beta} d\tau \int_{L^D} d^D\mathbf{r} \ \mathcal{L} \tag{6}$$

is the Euclidean action functional with Lagrangian density

$$\mathcal{L} = \bar{\psi}_s \left[ \hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \psi_s + \mathbf{g}\, \bar{\psi}_\uparrow\, \bar{\psi}_\downarrow\, \psi_\downarrow\, \psi_\uparrow \tag{7}$$

where $\mathbf{g}$ is the attractive strength ($\mathbf{g} < 0$) of the s-wave coupling.

[4] N. Nagaosa, Quantum Field Theory in Condensed Matter Physics (Springer, 1999)

# Quantum fluctuations in $D$-dimensions (II)

Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathcal{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(\mathbf{r}, \tau)$. In this way the effective Euclidean Lagrangian density reads

$$\mathcal{L}_e = \bar{\psi}_s \left[ \hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \psi_s + \bar{\Delta}\, \psi_\downarrow\, \psi_\uparrow + \Delta \bar{\psi}_\uparrow\, \bar{\psi}_\downarrow - \frac{|\Delta|^2}{\mathbf{g}} \,. \quad (8)$$

We investigate the effect of fluctuations of the gap field $\Delta(\mathbf{r}, t)$ around its mean-field value $\Delta_0$ which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r}, \tau) = \Delta_0 + \eta(\mathbf{r}, \tau) \,, \quad (9)$$

where $\eta(\mathbf{r}, \tau)$ is the complex field which describes pairing fluctuations.

# Quantum fluctuations in $D$-dimensions (III)

In particular, we are interested in the grand potential $\Omega$, given by

$$\Omega = -\frac{1}{\beta} \ln(\mathcal{Z}) \simeq -\frac{1}{\beta} \ln(\mathcal{Z}_{mf} \mathcal{Z}_g) = \Omega_{mf} + \Omega_B \; , \qquad (10)$$

where

$$\mathcal{Z}_{mf} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \exp\left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar} \right\} \qquad (11)$$

is the mean-field partition function and

$$\mathcal{Z}_g = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \mathcal{D}[\eta, \bar{\eta}] \; \exp\left\{ -\frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar} \right\} \qquad (12)$$

is the partition function of Gaussian pairing fluctuations.

# Quantum fluctuations in $D$-dimensions (IV)

One finds that in the gas of paired fermions there are two kinds of elementary excitations: fermionic single-particle excitations with energy

$$E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2} \, , \tag{13}$$

where $\Delta_0$ is the pairing gap, and bosonic collective excitations with energy

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda \, \frac{\hbar^2 q^2}{2m} + 2 \; m \; c_s^2\right)} \, , \tag{14}$$

where $\lambda$ is the first correction to the familiar low-momentum phonon dispersion $E_{col}(q) \simeq c_s \hbar q$ and $c_s$ is the sound velocity. Notice that both $\lambda$ and $c_s$ depend on the chemical potential $\mu$.

# Quantum fluctuations in $D$-dimensions (V)

Moreover, at the Gaussian level, the total grand potential reads

$$\Omega = \Omega_{mf} + \Omega_g \ , \tag{15}$$

where

$$\Omega_{mf} = \Omega_0 + \Omega_F^{(0)} + \Omega_F^{(T)} \tag{16}$$

is the mean-field grand potential with

$$\Omega_0 = -\frac{\Delta_0^2}{\mathbf{g}} L^D \tag{17}$$

the grand potential of the order parameter $\Delta_0$,

$$\Omega_F^{(0)} = -\sum_{\mathbf{k}} \left( E_{sp}(k) - \frac{\hbar^2 k^2}{2m} + \mu \right) \tag{18}$$

the zero-point energy of fermionic single-particle excitations,

$$\Omega_F^{(T)} = \frac{2}{\beta} \sum_{\mathbf{k}} \ln\left(1 + e^{-\beta\, E_{sp}(k)}\right) \tag{19}$$

the finite-temperature grand potential of the fermionic single-particle excitations.

# Quantum fluctuations in $D$-dimensions (VI)

The grand-potential of bosonic Gaussian fluctuations reads

$$\Omega_g = \Omega_{g,B}^{(0)} + \Omega_{g,B}^{(T)} , \tag{20}$$

where

$$\Omega_{g,B}^{(0)} = \frac{1}{2} \sum_{\mathbf{q}} E_{col}(q) \tag{21}$$

is the zero-point energy of bosonic collective excitations and

$$\Omega_{g,B}^{(T)} = \frac{1}{\beta} \sum_{\mathbf{q}} \ln\left(1 - e^{-\beta\, E_{col}(q)}\right) \tag{22}$$

is the finite-temperature grand potential of the bosonic collective excitations.
Both $\Omega_F^{(0)}$ and $\Omega_{g,B}^{(0)}$ are ultraviolet divergent in any dimension $D$ ($D = 1, 2, 3$) and the regularization of these divergent terms is complicated by the fact that one also must take into account the BCS-BEC crossover.

# New results for 2D BCS-BEC crossover (I)

In the analysis of the **two-dimensional attractive Fermi gas** one must remember that, contrary to the 3D case, 2D realistic interatomic attractive potentials have always a bound state. In particular[5], the binding energy $\epsilon_B > 0$ of two fermions can be written in terms of the positive 2D fermionic scattering length $a_s$ as

$$\epsilon_B = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{m{a_s}^2} , \tag{23}$$

where $\gamma = 0.577...$ is the Euler-Mascheroni constant. Moreover, the attractive (negative) interaction strength $\mathbf{g}$ of s-wave pairing is related to the binding energy $\epsilon_B > 0$ of a fermion pair in vacuum by the expression[6]

$$-\frac{1}{\mathbf{g}} = \frac{1}{2L^2} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 k^2}{2m} + \frac{1}{2}\epsilon_B} . \tag{24}$$

[5] C. Mora and Y. Castin, 2003, PRA **67**, 053615.

[6] M. Randeria, J-M. Duan, and L-Y. Shieh, PRL **62**, 981 (1989).

# New results for 2D BCS-BEC crossover (II)

In the **2D BCS-BEC crossover**, at zero temperature ($T = 0$) the mean-field grand potential $\Omega_{mf}$ can be written as[7] ($\epsilon_B > 0$)

$$\Omega_{mf} = -\frac{mL^2}{2\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2 \; . \tag{25}$$

Using

$$n = -\frac{1}{L^2}\frac{\partial\Omega_{mf}}{\partial\mu} \tag{26}$$

one immediately finds the chemical potential $\mu$ as a function of the number density $n = N/L^2$, i.e.

$$\mu = \frac{\pi\hbar^2}{m}n - \frac{1}{2}\epsilon_B \; . \tag{27}$$

In the BCS regime, where $\epsilon_B \ll \epsilon_F$ with $\epsilon_F = \pi\hbar^2 n/m$, one finds $\mu \simeq \epsilon_F > 0$ while in the BEC regime, where $\epsilon_B \gg \epsilon_F$ one has $\mu \simeq -\epsilon_B/2 < 0$.

[7] M. Randeria, J-M. Duan, and L-Y. Shieh, PRL **62**, 981 (1989).

# New results for 2D BCS-BEC crossover (III)

In the deep BEC regime of the **2D BCS-BEC crossover**, where the chemical potential $\mu$ becomes negative, performing regularization of zero-point fluctuations we have recently found[8] that the zero-temperature grand potential (including bosonic excitations) is

$$\Omega = -\frac{mL^2}{64\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2 \, \ln\left(\frac{\epsilon_B}{2(\mu + \frac{1}{2}\epsilon_B)}\right) . \qquad (28)$$

This is exactly Popov's equation of state of 2D Bose gas with chemical potential $\mu_B = 2(\mu + \epsilon_B/2)$ and mass $m_B = 2m$. In this way we have identified the two-dimensional scattering length $a_B$ of composite bosons as

$$\boxed{a_B = \frac{1}{2^{1/2}e^{1/4}} \, a_s \, .} \qquad (29)$$

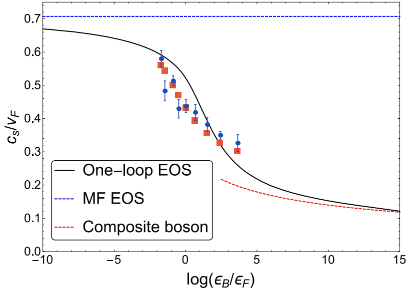
The value $a_B/a_s = 1/(2^{1/2}e^{1/4}) \simeq 0.551$ is in full agreement with $a_B/a_s = 0.55(4)$ obtained by Monte Carlo calculations[9].

[8] LS and F. Toigo, PRA **91**, 011604(R) (2015).
[9] G. Bertaina and S. Giorgini, PRL **106**, 110403 (2011).

# New results for 2D BCS-BEC crossover (IV)

At zero temperature we compare[10] the first sound velocity

$$c_s = \sqrt{\frac{n}{m}\frac{\partial \mu}{\partial n}} = \sqrt{-\frac{n}{m}\left(\frac{1}{L^2}\frac{\partial^2 \Omega(\mu)}{\partial \mu^2}\right)^{-1}} . \qquad (30)$$




with available experimental data[11] (blue circles and red squares).

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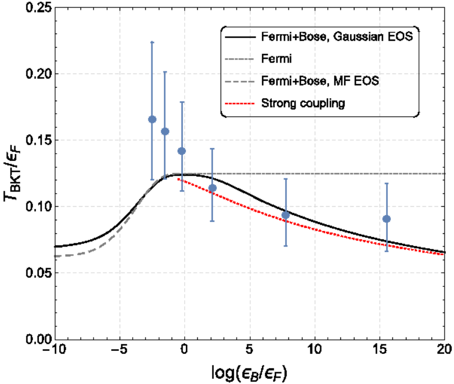
[10] G. Bighin and LS, PRB **93**, 014519 (2016).
[11] N. Luick, M.Sc. Thesis, Supervisors: E. Moritz and L. Mathey, University of Hamburg (2014).

# New results for 2D BCS-BEC crossover (V)

Also at finite temperature beyond-mean-field effects play a relevant role.



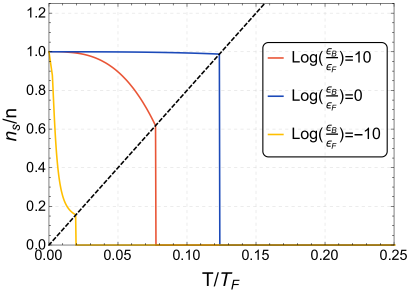

Theoretical predictions for the Berezinskii-Kosterlitz-Thouless critical temperature $T_{BKT}$ compared[12] to recent experimental observation[13] (circles with error bars).

[12] G. Bighin and LS, PRB **93**, 014519 (2016).
[13] P.A. Murthy et al., PRL **115**, 010401 (2015).

# New results for 2D BCS-BEC crossover (VII)

$T_{BKT}$ is determined with the Nelson-Kosterlitz condition[14]

$$k_B T_{BKT} = \frac{\hbar^2 \pi}{8m} n_s(T_{BKT}^-) \tag{31}$$




where $n_s(T)$ is the one-loop superfluid density, given by[15]

$$n_s(T) = n - \beta \int \frac{\mathrm{d}^2 k}{(2\pi)^2} k^2 \frac{e^{\beta E_{sp}(k)}}{(e^{\beta E_{sp}(k)}+1)^2} - \frac{\beta}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} q^2 \frac{e^{\beta E_{col}(q)}}{(e^{\beta E_{col}(q)}-1)^2} . \tag{32}$$

[14] D.R. Nelson and J.M. Kosterlitz, PRL **39**, 1201 (1977).
[15] G. Bighin and LS, PRB **93**, 014519 (2016).

# Conclusions

- The regularization of zero-point energy gives remarkable beyond-mean-field effects for composite bosons in the 2D BCS-BEC crossover at zero temperature:
  – logarithmic behavior of the equation of state
  – Bose-Bose scattering length $a_B$ vs Fermi-Fermi scattering length $a_s$
  – first sound velocity $c_s$
- Also at finite temperature beyond-mean-field effects become relevant in the strong-coupling regime of 2D BCS-BEC crossover:
  – critical temperature $T_{BKT}$
  – superfluid density $n_s$
  – second sound velocity $c_2$

# Acknowledgements

**Thank you for your attention!**

Main sponsors: CARIPARO Foundation, PRIN-MIUR, UNIPD.