Quantum fluctuations in two-dimensional Fermi superfluids

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Summary

- BCS-BEC crossover in 3D and 2D
- Quantum fluctuations in $D$-dimensions
- New results for 2D BCS-BEC crossover
- Conclusions
In 2004 the 3D BCS-BEC crossover has been observed with ultracold gases made of two-component fermionic $^{40}$K or $^6$Li alkali-metal atoms.\(^1\)

This crossover is obtained by using a Fano-Feshbach resonance to change the 3D s-wave scattering length $a_s$ of the inter-atomic potential

$$a_s = a_{bg} \left(1 + \frac{\Delta_B}{B - B_0}\right), \quad (1)$$

where $B$ is the external magnetic field.

\(^1\)C.A. Regal et al., PRL 92, 040403 (2004); M.W. Zwierlein et al., PRL 92, 120403 (2004); J. Kinast et al., PRL 92, 150402 (2004).
The 3D crossover from a BCS superfluid \((a_s < 0)\) to a BEC of molecular pairs \((a_s > 0)\) has been investigated experimentally around a Fano-Feshbach resonance, where the 3D s-wave scattering length \(a_s\) diverges, and it has been shown that the system is (meta)stable. The detection of quantized vortices under rotation\(^2\) has clarified that this dilute gas of ultracold atoms is superfluid. Usually the 3D BCS-BEC crossover is analyzed in terms of

\[
y = \frac{1}{k_F a_s}
\]  

(2)

the inverse scaled interaction strength, where \(k_F = (3\pi^2 n)^{1/3}\) is the Fermi wave number and \(n\) the total density. The system is dilute because \(r_e k_F \ll 1\), with \(r_e\) the effective range of the inter-atomic potential.

In the last few years also the 2D BEC-BEC crossover has been achieved\(^3\) with a quasi-2D Fermi gas of two-component \(^6\)Li atoms. In the 2D attractive system fermions always form biatomic molecules with bound-state energy

\[ \epsilon_B \simeq \frac{\hbar^2}{ma_s^2}, \]  

where \(a_s\) is the 2D s-wave scattering length. Both in 3D and 2D the fermionic single-particle spectrum is given by

\[ E_{sp}(k) = \sqrt{\left( \frac{\hbar^2 k^2}{2m} - \mu \right)^2 + \Delta^2}, \]  

where \(\Delta\) is the energy gap and \(\mu\) is the chemical potential: \(\mu > 0\) corresponds to the BCS regime while \(\mu < 0\) corresponds to the BEC regime. Moreover, in the deep BEC regime \(\mu \rightarrow -\epsilon_B/2\).

Quantum fluctuations in $D$-dimensions (I)

We adopt the formalism of functional integration\textsuperscript{4}. The partition function $\mathcal{Z}$ of the uniform system with fermionic fields $\psi_s(\mathbf{r}, \tau)$ at temperature $T$, in a $D$-dimensional volume $L^D$, and with chemical potential $\mu$ reads

$$
\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{S}{\hbar} \right\},
$$

(5)

where ($\beta \equiv 1/(k_B T)$ with $k_B$ Boltzmann’s constant)

$$
S = \int_0^{\hbar \beta} d\tau \int_{L^D} d^D \mathbf{r} \mathcal{L}
$$

(6)

is the Euclidean action functional with Lagrangian density

$$
\mathcal{L} = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow
$$

(7)

where $g$ is the attractive strength ($g < 0$) of the s-wave coupling.

\textsuperscript{4}N. Nagaosa, Quantum Field Theory in Condensed Matter Physics (Springer, 1999)
Quantum fluctuations in $D$-dimensions (II)

Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathcal{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(r, \tau)$. In this way the effective Euclidean Lagrangian density reads

$$\mathcal{L}_{e} = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + \bar{\Delta} \psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow - \frac{|\Delta|^2}{g}. \quad (8)$$

We investigate the effect of fluctuations of the gap field $\Delta(r, t)$ around its mean-field value $\Delta_0$ which may be taken to be real. For this reason we set

$$\Delta(r, \tau) = \Delta_0 + \eta(r, \tau), \quad (9)$$

where $\eta(r, \tau)$ is the complex field which describes pairing fluctuations.
In particular, we are interested in the grand potential \( \Omega \), given by

\[
\Omega = -\frac{1}{\beta} \ln (Z) \simeq \frac{1}{\beta} \ln (Z_{mf} Z_g) = \Omega_{mf} + \Omega_B ,
\]

where

\[
Z_{mf} = \int \mathcal{D} [\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar} \right\}
\]

is the mean-field partition function and

\[
Z_g = \int \mathcal{D} [\psi_s, \bar{\psi}_s] \mathcal{D} [\eta, \bar{\eta}] \exp \left\{ -\frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar} \right\}
\]

is the partition function of Gaussian pairing fluctuations.
One finds that in the gas of paired fermions there are two kinds of elementary excitations: fermionic single-particle excitations with energy

$$E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2}, \quad (13)$$

where $\Delta_0$ is the pairing gap, and bosonic collective excitations with energy

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m} \left(\lambda \frac{\hbar^2 q^2}{2m} + 2m c_s^2\right)}, \quad (14)$$

where $\lambda$ is the first correction to the familiar low-momentum phonon dispersion $E_{col}(q) \simeq c_s \hbar q$ and $c_s$ is the sound velocity. Notice that both $\lambda$ and $c_s$ depend on the chemical potential $\mu$. 


Quantum fluctuations in $D$-dimensions (V)

Moreover, at the Gaussian level, the total grand potential reads

$$\Omega = \Omega_{mf} + \Omega_g,$$  \hspace{1cm} (15)

where

$$\Omega_{mf} = \Omega_0 + \Omega_F^{(0)} + \Omega_F^{(T)}$$  \hspace{1cm} (16)

is the mean-field grand potential with

$$\Omega_0 = -\frac{\Delta_0^2}{g}L^D$$  \hspace{1cm} (17)

the grand potential of the order parameter $\Delta_0$,

$$\Omega_F^{(0)} = -\sum_k \left( E_{sp}(k) - \frac{\hbar^2 k^2}{2m} + \mu \right)$$  \hspace{1cm} (18)

the zero-point energy of fermionic single-particle excitations,

$$\Omega_F^{(T)} = \frac{2}{\beta} \sum_k \ln \left( 1 + e^{-\beta E_{sp}(k)} \right)$$  \hspace{1cm} (19)

the finite-temperature grand potential of the fermionic single-particle excitations.
Quantum fluctuations in $D$-dimensions (VI)

The grand-potential of bosonic Gaussian fluctuations reads

$$\Omega_g = \Omega^{(0)}_{g,B} + \Omega^{(T)}_{g,B},$$

(20)

where

$$\Omega^{(0)}_{g,B} = \frac{1}{2} \sum_q E_{\text{col}}(q)$$

(21)

is the zero-point energy of bosonic collective excitations and

$$\Omega^{(T)}_{g,B} = \frac{1}{\beta} \sum_q \ln (1 - e^{-\beta E_{\text{col}}(q)})$$

(22)

is the finite-temperature grand potential of the bosonic collective excitations.

Both $\Omega^{(0)}_F$ and $\Omega^{(0)}_{g,B}$ are ultraviolet divergent in any dimension $D$ ($D = 1, 2, 3$) and the regularization of these divergent terms is complicated by the fact that one also must take into account the BCS-BEC crossover.
In the analysis of the **two-dimensional attractive Fermi gas** one must remember that, contrary to the 3D case, **2D realistic interatomic attractive potentials have always a bound state**. In particular, the binding energy $\epsilon_B > 0$ of two fermions can be written in terms of the positive 2D fermionic scattering length $a_s$ as

$$\epsilon_B = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{ma_s^2},$$

(23)

where $\gamma = 0.577...$ is the Euler-Mascheroni constant. Moreover, the attractive (negative) interaction strength $g$ of s-wave pairing is related to the binding energy $\epsilon_B > 0$ of a fermion pair in vacuum by the expression

$$-\frac{1}{g} = \frac{1}{2L^2} \sum_k \frac{1}{\frac{\hbar^2k^2}{2m} + \frac{1}{2}\epsilon_B}.$$

(24)

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In the 2D BCS-BEC crossover, at zero temperature ($T = 0$) the mean-field grand potential $\Omega_{mf}$ can be written as\(^7\) ($\epsilon_B > 0$)

$$\Omega_{mf} = -\frac{mL^2}{2\pi\hbar^2} (\mu + \frac{1}{2} \epsilon_B)^2 . \quad (25)$$

Using

$$n = -\frac{1}{L^2} \frac{\partial \Omega_{mf}}{\partial \mu} \quad (26)$$

one immediately finds the chemical potential $\mu$ as a function of the number density $n = N/L^2$, i.e.

$$\mu = \frac{\pi\hbar^2}{m} n - \frac{1}{2} \epsilon_B . \quad (27)$$

In the BCS regime, where $\epsilon_B \ll \epsilon_F$ with $\epsilon_F = \pi\hbar^2 n/m$, one finds $\mu \simeq \epsilon_F > 0$ while in the BEC regime, where $\epsilon_B \gg \epsilon_F$ one has $\mu \simeq -\epsilon_B / 2 < 0$.

\(^7\)M. Randeria, J-M. Duan, and L-Y. Shieh, PRL 62, 981 (1989).
In the deep BEC regime of the 2D BCS-BEC crossover, where the chemical potential $\mu$ becomes negative, performing regularization of zero-point fluctuations we have recently found\textsuperscript{8} that the zero-temperature grand potential (including bosonic excitations) is

$$
\Omega = -\frac{mL^2}{64\pi \hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2 \ln \left( \frac{\epsilon_B}{2(\mu + \frac{1}{2}\epsilon_B)} \right). \quad (28)
$$

This is exactly Popov’s equation of state of 2D Bose gas with chemical potential $\mu_B = 2(\mu + \epsilon_B/2)$ and mass $m_B = 2m$. In this way we have identified the two-dimensional scattering length $a_B$ of composite bosons as

$$
a_B = \frac{1}{2^{1/2}e^{1/4}} a_s. \quad (29)
$$

The value $a_B/a_s = 1/(2^{1/2}e^{1/4}) \approx 0.551$ is in full agreement with $a_B/a_s = 0.55(4)$ obtained by Monte Carlo calculations\textsuperscript{9}.

\textsuperscript{8}LS and F. Toigo, PRA 91, 011604(R) (2015).
At zero temperature we compare\(^{10}\) the first sound velocity

\[ c_s = \sqrt{\frac{n}{m} \frac{\partial \mu}{\partial n}} = \sqrt{-\frac{n}{m} \left( \frac{1}{L^2} \frac{\partial^2 \Omega(\mu)}{\partial \mu^2} \right)^{-1}}. \]  \hspace{1cm} (30)

with available experimental data\(^{11}\) (blue circles and red squares).

\(^{10}\)G. Bighin and LS, PRB 93, 014519 (2016).

New results for 2D BCS-BEC crossover (V)

Also at finite temperature beyond-mean-field effects play a relevant role.

Theoretical predictions for the Berezinskii-Kosterlitz-Thouless critical temperature $T_{BKT}$ compared$^{12}$ to recent experimental observation$^{13}$ (circles with error bars).

$^{12}$G. Bighin and LS, PRB 93, 014519 (2016).
$^{13}$P.A. Murthy et al., PRL 115, 010401 (2015).
New results for 2D BCS-BEC crossover (VII)

$T_{BKT}$ is determined with the Nelson-Kosterlitz condition\textsuperscript{14}

\[ k_B T_{BKT} = \frac{\hbar^2 \pi}{8m} n_s(T_{BKT}) \] (31)

where $n_s(T)$ is the one-loop superfluid density, given by\textsuperscript{15}

\[ n_s(T) = n - \beta \int \frac{d^2 k}{(2\pi)^2} k^2 \frac{e^{\beta E_{sp}(k)}}{(e^{\beta E_{sp}(k)} + 1)^2} - \frac{\beta}{2} \int \frac{d^2 q}{(2\pi)^2} q^2 \frac{e^{\beta E_{col}(q)}}{(e^{\beta E_{col}(q)} - 1)^2}. \] (32)


\textsuperscript{15}G. Bighin and LS, PRB \textbf{93}, 014519 (2016).
The regularization of zero-point energy gives remarkable beyond-mean-field effects for composite bosons in the 2D BCS-BEC crossover at zero temperature:
- logarithmic behavior of the equation of state
- Bose-Bose scattering length $a_B$ vs Fermi-Fermi scattering length $a_s$
- first sound velocity $c_s$

Also at finite temperature beyond-mean-field effects become relevant in the strong-coupling regime of 2D BCS-BEC crossover:
- critical temperature $T_{BKT}$
- superfluid density $n_s$
- second sound velocity $c_2$
Thank you for your attention!

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