BCS-BEC crossover with Rashba and Dresselhaus couplings

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Summary

- BCS-BEC crossover
- Artificial spin-orbit coupling
- Mean-field approach
- Singlet and triplet condensate
- Results with Rashba coupling
- Including Dresselhaus coupling
- Conclusions
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In 2004 the BCS-BEC crossover has been observed with ultracold gases made of fermionic $^{40}\text{K}$ and $^{6}\text{Li}$ alkali-metal atoms.¹

This crossover is obtained by changing (with a Feshbach resonance) the s-wave scattering length $a_s$ of the inter-atomic potential:

$- a_s \rightarrow 0^-$ (BCS regime of weakly-interacting Cooper pairs)

$- a_s \rightarrow \pm\infty$ (unitarity limit of strongly-interacting Cooper pairs)

$- a_s \rightarrow 0^+$ (BEC regime of bosonic dimers)

¹C.A. Regal et al., PRL 92, 040403 (2004); M.W. Zwierlein et al., PRL 92, 120403 (2004); M. Bartenstein, A. Altmeyer et al., PRL 92, 120401 (2004); J. Kinast et al., PRL 92, 150402 (2004).
The crossover from a BCS superfluid ($a_s < 0$) to a BEC of molecular pairs ($a_s > 0$) has been investigated experimentally around a Feshbach resonance, where the s-wave scattering length $a_s$ diverges, and it has been shown that the system is (meta)stable. The detection of quantized vortices under rotation\(^2\) has clarified that this dilute and ultracold gas of Fermi atoms is superfluid. Usually the BCS-BEC crossover is analyzed in terms of

$$y = \frac{1}{k_F a_s}$$

the inverse scaled interaction strength, where $k_F = (\frac{3\pi^2 n}{2})^{1/3}$ is the Fermi wave number and $n$ the total density. The system is dilute because $r_e k_F \ll 1$, with $r_e$ the effective range of the inter-atomic potential.

In 2011 and 2012 artificial spin-orbit coupling has been experimentally imposed on both bosonic\textsuperscript{3} and fermionic\textsuperscript{4} atomic gases. The single-particle Hamiltonian $\hat{h}_{sp}$ with both Rashba and Dresselhaus spin-orbit couplings reads

$$\hat{h}_{sp} = \frac{\hat{p}^2}{2m} + v_R (\hat{\sigma}_x \hat{p}_y - \hat{\sigma}_y \hat{p}_x) + v_D (\hat{\sigma}_x \hat{p}_y + \hat{\sigma}_y \hat{p}_x),$$

(2)

with $\hat{p}^2 = -\hbar^2 \nabla^2$, $\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$, $\hat{p}_y = -i\hbar \frac{\partial}{\partial y}$, $v_R$ and $v_D$ the Rashba and Dresselhaus coupling constant, respectively, and

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$


\textsuperscript{4}P. Wang et al., PRL 109, 095301 (2012); L.W. Cheuk et al., PRL 109, 095302 (2012).
The partition function \( Z \) of the uniform two-spin-component Fermi system at temperature \( T \), in a volume \( V \), and with chemical potential \( \mu \) can be written in terms of a functional integral as

\[
Z = \int D[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{1}{\hbar} S \right\},
\]

where

\[
S = \int_0^{\hbar \beta} d\tau \int_V d^3r \; \mathcal{L}
\]

is the Euclidean action functional and \( \mathcal{L} \) is the Euclidean Lagrangian density, given by

\[
\mathcal{L} = (\bar{\psi}_\uparrow, \bar{\psi}_\downarrow) \left[ \hat{\hbar} \partial_\tau + \hat{\hbar}_{sp} - \mu \right] \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} + g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow
\]

with \( g \) is the strength of the s-wave coupling (\( g < 0 \) in the BCS regime). Notice that \( \beta = 1/(k_B T) \) with \( k_B \) the Boltzmann constant. In the rest of the seminar we shall use units such that \( \hbar = m = k_B = 1 \).
The Lagrangian density $\mathcal{L}$ is quartic in the fermionic fields $\psi_s$, but one can reduce the problem to a quadratic Lagrangian density by introducing an auxiliary complex scalar field $\Delta(r, \tau)$ via Hubbard-Stratonovich transformation$^5$, which gives

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \mathcal{D}[\Delta, \bar{\Delta}] \exp \left\{ -S_e \right\}, \quad (6)$$

where

$$S_e = \int_0^{1/T} d\tau \int_V d^3r \mathcal{L}_e \quad (7)$$

and the (exact) effective Euclidean Lagrangian density $\mathcal{L}_e$ reads

$$\mathcal{L}_e = (\bar{\psi}_\uparrow, \bar{\psi}_\downarrow) \left[ \partial_\tau + \hat{h}_{sp} - \mu \right] \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} + \bar{\Delta} \psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow - \frac{|\Delta|^2}{g}. \quad (8)$$

It is a standard procedure to integrate out the quadratic fermionic fields and to get a new formally-exact effective action $S_{\text{eff}}$ which depends only on the auxiliary field $\Delta(r, \tau)$. In this way we obtain

$$Z = \int \mathcal{D}[\Delta, \bar{\Delta}] \exp \{-S_{\text{eff}}\} ,$$

where

$$S_{\text{eff}} = -Tr[\ln \left( G^{-1} \right)] - \int_0^{1/T} d\tau \int_V d^3r \frac{|\Delta|^2}{g}$$

with $\gamma(\hat{p}) = v_R(\hat{p}_y + i\hat{p}_x) + v_D(\hat{p}_y - i\hat{p}_x)$ and

$$G^{-1} = \begin{pmatrix}
\partial_\tau + \frac{\hat{p}_x^2}{2} - \mu & \Delta & \gamma(\hat{p}) & 0 \\
\Delta & \partial_\tau - \frac{\hat{p}_x^2}{2} + \mu & 0 & -\gamma(-\hat{p}) \\
\bar{\gamma}(\hat{p}) & 0 & \partial_\tau + \frac{\hat{p}_x^2}{2} - \mu & -\Delta \\
0 & -\bar{\gamma}(-\hat{p}) & -\bar{\Delta} & \partial_\tau - \frac{\hat{p}_x^2}{2} + \mu \\
\end{pmatrix}$$
For a uniform Fermi superfluid within the simplest mean-field approximation one has a constant and real gap parameter, i.e. \( \Delta(r, \tau) = \Delta \), and the partition function becomes\(^6\)

\[
Z_{mf} = \exp \left\{ -S_{mf} \right\} = \exp \left\{ -\frac{\Omega_{mf}}{T} \right\},
\]

where

\[
S_{mf} = \frac{\Omega_{mf}}{T} = -\sum_k \left[ \sum_{j=1}^{4} \ln \left( 1 + e^{-E_{k,j}/T} \right) - \frac{\xi_k}{T} \right] - \frac{V}{T} \frac{\Delta^2}{g},
\]

with \( \xi_k = k^2/2 - \mu \), \( \gamma_k = v_R(k_y + ik_x) + v_D(k_y - ik_x) \), and

\[
E_{k,1} = \sqrt{(\xi_k - |\gamma_k|)^2 + \Delta^2}, \quad E_{k,3} = -E_{k,1}, \quad E_{k,4} = -E_{k,2}, \quad E_{k,2} = \sqrt{(\xi_k + |\gamma_k|)^2 + \Delta^2},
\]

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\(^6\)L. Dell’Anna, G. Mazzarella, L.S., PRA 84, 033633 (2011).
Mean-field approach (V)

The constant and real gap parameter $\Delta$ is obtained from

$$\frac{\partial S_{mf}}{\partial \Delta} = 0 ,$$

which gives the gap equation

$$-\frac{1}{g} = \frac{1}{V} \sum_k \sum_{j=1,2} \frac{\tanh(E_{k,j}/2T)}{4E_{k,j}} .$$

(17)

The integral on the right side of this equation is formally divergent. However, expressing the bare interaction strength $g$ in terms of the physical scattering length $a_s$ with the formula

$$-\frac{1}{g} = -\frac{1}{4\pi a_s} + \frac{1}{V} \sum_k \frac{1}{k^2} ,$$

(18)

one obtains the regularized gap equation

$$-\frac{1}{4\pi a_s} = \frac{1}{V} \sum_k \left[ \sum_{j=1,2} \frac{\tanh(E_{k,j}/2T)}{4E_{k,j}} - \frac{1}{k^2} \right] .$$

(19)

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From the thermodynamic formula

\[ N = - \left( \frac{\partial \Omega_{mf}}{\partial \mu} \right)_{V,T} \]  

(20)

one obtains also the **equation for the number of particles**\(^9\)

\[ N = \sum_k \left( 1 - \frac{\xi_k - |\gamma_k|}{2E_{k,1}} \tanh \left( \frac{E_{k,1}}{2T} \right) - \frac{\xi_k + |\gamma_k|}{2E_{k,2}} \tanh \left( \frac{E_{k,2}}{2T} \right) \right). \]  

(21)

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Singlet and triplet condensation (I)

In a Fermi system the largest eigenvalue $N_C$ of the two-body density matrix gives the number of correlated fermion pairs which have their center of mass with zero linear momentum.\(^\text{10}\) This condensed number of pairs is given by

$$N_C = N_0 + N_1,$$

where

$$N_0 = \int d^3r \, d^3r' \left[ |\langle \psi_\downarrow(r) \, \psi_\uparrow(r') \rangle|^2 + |\langle \psi_\uparrow(r) \, \psi_\downarrow(r') \rangle|^2 \right]$$

(23)

is the condensed number of pairs in the spin 0 state ($m_s = 0$), while

$$N_1 = \int d^3r \, d^3r' \left[ |\langle \psi_\uparrow(r) \, \psi_\uparrow(r') \rangle|^2 + |\langle \psi_\downarrow(r) \, \psi_\downarrow(r') \rangle|^2 \right].$$

(24)

is the condensed number of pairs in the spin 1 state ($|m_s| = 1$).

Singlet and triplet condensation (II)

In our superfluid Fermi system with spin-orbit coupling we obtain\(^{11}\)

\[
N_0 = \frac{\Delta^2}{4} \sum_k \left( \frac{1}{2E_{k,1}} \tanh \left( \frac{E_{k,1}}{2T} \right) + \frac{1}{2E_{k,2}} \tanh \left( \frac{E_{k,2}}{2T} \right) \right)^2. \tag{25}
\]

and

\[
N_1 = \frac{\Delta^2}{4} \sum_k \left( \frac{1}{2E_{k,1}} \tanh \left( \frac{E_{k,1}}{2T} \right) - \frac{1}{2E_{k,2}} \tanh \left( \frac{E_{k,2}}{2T} \right) \right)^2. \tag{26}
\]

Notice that in the absence of spin-orbit coupling \((v_R = v_D = 0)\) one has \(E_{k,1} = E_{k,2}\) from which one gets \(N_1 = 0\), and consequently the condensate number of Cooper pairs in the triplet state is zero.

\(^{11}\)L. Dell’Anna, G. Mazzarella, L.S., PRA 84, 033633 (2011).
We are interested in the low temperature regime where the condensate fraction can be quite large. Quantitatively we restrict our study to the zero temperature limit ($T=0$). In the equations above we have therefore simply $\tanh(E_{k,j}/2T) \to 1$. In this way the regularized gap equation is given by

$$-\frac{1}{4\pi a_s} = \frac{1}{V} \sum_k \left[ \sum_{j=1,2} \frac{1}{4E_{k,j}} - \frac{1}{k^2} \right],$$

(27)

while the number equation reads

$$N = \sum_k \left(1 - \frac{\xi_k - |\gamma_k|}{2E_{k,1}} - \frac{\xi_k + |\gamma_k|}{2E_{k,2}} \right).$$

(28)
Similarly, we obtain for the spin 0 condensate number

\[ N_0 = \frac{\Delta^2}{4} \sum_k \left( \frac{1}{2E_{k,1}} + \frac{1}{2E_{k,2}} \right)^2. \]  

(29)

and for the spin 1 condensate number

\[ N_1 = \frac{\Delta^2}{4} \sum_k \left( \frac{1}{2E_{k,1}} - \frac{1}{2E_{k,2}} \right)^2. \]  

(30)

From the previous equations one can calculate the chemical potential \( \mu \), the energy gap \( \Delta \), and also the condensate fractions \( N_0/(N/2) \) and \( N_1/(N/2) \), as a function of the scaled interaction strength \( y = 1/(k_F a_S) \).

**Note:** We now show the results obtained for \( \nu_D = 0 \), i.e. when **only** Rashba spin-orbit coupling is active.
Scaled chemical potential $\mu/\epsilon_F$ as a function of the adimensional interaction strength $y = 1/(k_F a_s)$ for different values of the scaled Rashba velocity: $v_R/v_F = 0$ (solid line), $v_R/v_F = 0.7$ (long-dashed line), $v_R/v_F = 1$ (short-dashed line), $v_R/v_F = 1.4$ (dotted line), $v_R/v_F = 2$ (dashed-dotted line). Here $\epsilon_F = v_F^2/2$ is the Fermi energy and $v_F = (3\pi^2 n)^{1/3}$ is the Fermi velocity.
Scaled energy gap $\Delta/\epsilon_F$ as a function of the adimensional interaction strength $y = 1/(k_F a_s)$ for different values of the scaled Rashba velocity: $v_R/v_F = 0$ (solid line), $v_R/v_F = 0.7$ (long-dashed line), $v_R/v_F = 1$ (short-dashed line), $v_R/v_F = 1.4$ (dotted line), $v_R/v_F = 2$ (dashed-dotted line). Here $\epsilon_F = v_F^2/2$ is the Fermi energy and $v_F = (3\pi^2 n)^{1/3}$ is the Fermi velocity.
Spin 0 condensate fraction $n_0/(n/2)$ (upper curves) and spin 1 condensate fraction $n_1/(n/2)$ (lower curves) as a function of the adimensional interaction strength $y = 1/(k_F a_s)$ for different values of the scaled Rashba velocity: $v_R/v_F = 0$ (solid line), $v_R/v_F = 0.7$ (long-dashed line), $v_R/v_F = 1$ (short-dashed line), $v_R/v_F = 1.4$ (dotted line), $v_R/v_F = 2$ (dashed-dotted line). Here $v_F = (3\pi^2 n)^{1/3}$ is the Fermi velocity.
We now consider also the Dresselhaus coupling, i.e. $\nu_D \neq 0$. For simplicity we set\(^\text{12}\)

\[
\nu_R = \nu \cos (\theta),
\]

\[
\nu_D = \nu \sin (\theta),
\]

where $\theta$ is the mixing angle.

\(^{12}\)L. Dell’Anna, G. Mazzarella, L.S., PRA 86, 053632 (2012).
Spin 0 condensate fraction $n_0/(n/2)$ as a function of the adimensional interaction strength $y = 1/(k_F a_s)$ for three values of $v$ and two values of $\theta$: $\theta = 0$ means $v_R = v$ and $v_D = 0$ (red solid curves), while $\theta = \pi/4$ means $v_R = v_D = v/\sqrt{2}$ (blue dotted curves). $v$ is in units of the Fermi velocity $v_F = (3\pi^2 n)^{1/3}$. 
Including Dresselhaus coupling (III)

Spin 1 condensate fraction $n_1/(n/2)$ as a function of the adimensional interaction strength $y = 1/(k_F a_s)$ for three values of $v$ and two values of $\theta$: $\theta = 0$ means $v_R = v$ and $v_D = 0$ (red solid curves), while $\theta = \pi/4$ means $v_R = v_D = v/\sqrt{2}$ (blue dotted curves). $v$ is in units of the Fermi velocity $v_F = (3\pi^2 n)^{1/3}$. 
Conclusions

- With a large spin-orbit coupling \( v \) the chemical potential \( \mu \) is negative for any interaction strength \( y \).
- In the deep BCS regime \( (y \ll -1) \) and small \( v \) we get the analytical result
  \[
  \mu = \epsilon_F - v^2 .
  \]

A finite condensate fraction \( n_1/n \) of triplet pairs appears due to the spin-orbit coupling, in addition to the familiar condensate fraction \( n_0/n \) of singlet pairs.

In the case where Rashba is equal to Dresselhaus, i.e. \( \nu_R = \nu_D = \nu / \sqrt{2} \) with \( \theta = \pi/4 \), we find the following formulas (for any \( y \))

\[
\mu(\nu, \theta = \pi/4) = \mu(\nu = 0) - v^2 ,
\]
\[
\Delta(\nu, \theta = \pi/4) = \Delta(\nu = 0) .
\]
\[
n_c(\nu, \theta = \pi/4) = n_c(\nu = 0) ,
\]

where \( n_c = n_0 + n_1 \) is the total condensate density.
THANK YOU FOR YOUR ATTENTION!

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