Solitons and vortices in Bose-Einstein condensates with finite-range interaction

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Summary

- Bose-Einstein condensates and Hartree equation
- Finite-range potential and modified Gross-Pitaevskii equation
- Solitons with the 1D MGPE
- Vortices with the 3D MGPE
- Conclusions

Bose-Einstein condensates and Hartree equation (I)

Let us consider a system of N identical bosonic particles of mass m described by the many-body Hamiltonian

$$\hat{H} = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m} \nabla_i^2 + U(\mathbf{r}_i) \right] + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} V(\mathbf{r} - \mathbf{r}') \ , \qquad (1)$$

where $U(\mathbf{r})$ is the external trapping potential and $V(\mathbf{r})$ is the two-body interaction potential.
Let us assume that all the bosonic particles are in a pure Bose-Einstein condensate, characterized by the same single-particle wave function $\psi(\mathbf{r})$. Moreover, let us assume that the symmetric many-body wavefunction of the system is given by

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{N-1}, \mathbf{r}_N) = \psi(\mathbf{r}_1)\, \psi(\mathbf{r}_2) \ldots \psi(\mathbf{r}_{N-1})\, \psi(\mathbf{r}_N) \ , \qquad (2)$$

which is clearly invariant exchanging two coordinates.

Bose-Einstein condensates and Hartree equation (II)

The expectation value of $\hat{H}$ with respect to Ψ gives

$$\begin{aligned} E[\psi(\mathbf{r})] = \langle \Psi | \hat{H} | \Psi \rangle &= N \int d^3\mathbf{r}\, \psi^*(\mathbf{r}) \left[-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) \right] \psi(\mathbf{r}) \\ &+ \frac{1}{2} N(N-1) \int d^3\mathbf{r} d^3\mathbf{r}' \, |\psi(\mathbf{r})|^2 V(\mathbf{r}-\mathbf{r}') |\psi(\mathbf{r}')|^2 \, . \end{aligned} \tag{3}$$

Extremizing this energy functional with respect to $\psi(\mathbf{r})$, and taking also into account the following constraint of normalization

$$\int d^3\mathbf{r} \, |\psi(\mathbf{r})|^2 = 1 \, , \tag{4}$$

one immediately finds

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}) + (N-1) \int d^3\mathbf{r}' \, V(\mathbf{r}-\mathbf{r}') |\psi(\mathbf{r}')|^2 \right] \psi(\mathbf{r}) = \mu \, \psi(\mathbf{r}) \, , \tag{5}$$

which is the so-called Hartree equation for identical bosons and μ is the chemical potential (Lagrange multiplier) of the system.

Finite-range potential and MGPE (I)

The Hartree equation is a nonlinear Schrödinger equation with a nonlocal nonlinearity. However, to describe Bose-Einstein condensates in ultracold and dilute atomic gases, the inter-atomic potential $V(\mathbf{r})$ is usually approximated by the zero-range Fermi pseudo-potential, namely

$$V(\mathbf{r}-\mathbf{r}') = g_0\,\delta(\mathbf{r}-\mathbf{r}')\ , \tag{6}$$

where $\delta(\mathbf{r})$ is the Dirac delta function and g_0 is the strength of the zero-range interaction ($g_0 > 0$ repulsion, $g_0 < 0$ attraction).
In this way the Hartee equation for bosons becomes

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + (N-1)g_0|\psi(\mathbf{r})|^2\right]\psi(\mathbf{r}) = \mu\,\psi(\mathbf{r})\ , \tag{7}$$

which is the familiar Gross-Pitaevskii equation[1] (GPE), i.e. a nonlinear Schrödinger equation with cubic nonlinearity.

[1] E.P. Gross, Nuovo Cimento **20**, 454 (1961); L. Pitaevskii, Sov. Phys. JETP **13**, 451 (1961).

Finite-range potential and MGPE (II)

In the Hartree equation for bosons there is the mean-field nonlocal potential

$$U_{mf}(\mathbf{r}) = \int d^3\mathbf{r}' \, |\psi(\mathbf{r}')|^2 \, V(\mathbf{r}-\mathbf{r}') \; . \tag{8}$$

Substituting $\mathbf{r}' = \mathbf{r} + \mathbf{s}$ in the right side of Eq. (8) and expanding $\psi(\mathbf{r}+\mathbf{s})$ in powers of $\mathbf{s}$ one gets at the second order[2]

$$U_{mf}(\mathbf{r}) = g_0|\psi(\mathbf{r})|^2 + g_2\nabla^2|\psi(\mathbf{r})|^2 \; , \tag{9}$$

where

$$g_0 = \int d^3\mathbf{s}\, V(s) = \tilde{V}(0) \quad \text{and} \quad g_2 = \frac{1}{2}\int d^3\mathbf{s}\, s^2 \, V(s) = -\frac{1}{2}\tilde{V}''(0) \tag{10}$$

with $\tilde{V}(q)$ the Fourier transform of $V(r)$.

[2] A. Parola, L. Salasnich, L. Reatto, Phys. Rev. A **57**, R3180 (1998); J.J. Garcia-Ripoll, V.V. Konotop, B. A. Malomed, and V.M. Perez-Garcia, Mathematics and Computers in Simulation **62**, 21 (2003).

Finite-range potential and MGPE (III)

Thus, by using Eq. (9) the Hartree equation for bosons becomes

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + (N-1)g_0|\psi(\mathbf{r})|^2 + g_2(N-1)\nabla^2|\psi(\mathbf{r})|^2\right]\psi(\mathbf{r}) = \mu\,\psi(\mathbf{r}) \ , \tag{11}$$

which is a modified Gross-Pitaevskii equation (MGPE).
The MGPE can also be obtained from the Hartree equation of bosons by using the two-body pseudo-potential[3]

$$V(\mathbf{r}) = g_0\,\delta(\mathbf{r}) + \frac{g_2}{2}\left(\overleftarrow{\nabla}^2\delta(\mathbf{r}) + \delta(r)\,\overrightarrow{\nabla}^2\right) \ , \tag{12}$$

where the Laplace operators $\overleftarrow{\nabla}^2$ and $\overrightarrow{\nabla}^2$ act respectively on left and right functions giving the appropriate symmetrization on the effective local potential.

[3] R. Roth and H. Feldmeier, Phys. Rev. A **64**, 043603 (2001); H. Fu, Y. Wang, and B. Gao, Phys. Rev. A **67**, 053612 (2003).

Solitons with the 1D MGPE (I)

In 2003 Garcia-Ripoll, Konotop, Malomed, and Perez-Garcia[4] studied spherically-symmetric solutions of the MGPE with a harmonic trapping potential

$$U(\mathbf{r}) = \frac{1}{2} m\omega^2 r^2 \ . \qquad (13)$$

In the case $g_0 < 0$ they found solitonic solutions of the MGPE which are stabilized against collapse by the finite-range parameter g_2, as shown also by Parola, Salasnich, Reatto solving the Hartree equation[5].

In 2014 Veksler, Fishman, and Ketterle[6] showed that for a **quasi-1D Bose-Einstein condensate** made of dilute atoms the MGPE gives rise to small but observable corrections to the GPE.

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[4] J.J. Garcia-Ripoll, V.V. Konotop, B. A. Malomed, and V.M. Perez-Garcia, Mathematics and Computers in Simulation **62**, 21 (2003).

[5] A. Parola, L. Salasnich, L. Reatto, Phys. Rev. A **57**, R3180 (1998).

[6] H. Veksler, S. Fishman, and W. Ketterle, Phys. Rev. A **90**, 023620 (2014).

Solitons with the 1D MGPE (II)

The 1D MGPE is obtained from the 3D MGPE assuming the following trapping potential

$$U(\mathbf{r}) = \frac{1}{2} m\omega_\perp^2 (x^2 + y^2) + W(z) \qquad (14)$$

and the following wavefunction

$$\psi(\mathbf{r}) = \frac{e^{-(x^2+y^2)/(2a_\perp^2)}}{\pi^{1/2} a_\perp} \, \phi(z) \; , \qquad (15)$$

where $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$ is the characteristic length of the transverse harmonic confinement and $W(z)$ is a generic axial potential. The corresponding 1D MGPE for the axial wavefunction $\phi(z)$ reads

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + W(z) + \gamma_0 |\phi(z)|^2 + \frac{1}{2} \gamma_2 \frac{\partial^2}{\partial z^2} |\phi(z)|^2 \right] \phi(z) = \tilde{\mu} \, \phi(z) \; , \qquad (16)$$

where $\gamma_0 = (g_0 - g_2/a_\perp^2)/(2\pi a_\perp^2)$, $\gamma_2 = g_2/(2\pi a_\perp^2)$ and $\tilde{\mu} = \mu - \hbar\omega_\perp$.

Solitons with the 1D MGPE (III)

Setting for simplicity $\hbar = m = 1$, $W(z) = 0$, and assuming a real wavefunction $\psi(z)$, the 1D MGPE reads

$$-\frac{1}{2}\phi'' + \gamma_0\phi^2 + \frac{1}{2}\gamma_2(\phi^2)''\phi = \mu\,\phi\ . \tag{17}$$

If $\gamma_0 > 0$ and $\gamma_2 < 1/2$ this equation admits exact dark soliton solutions $\phi(z)$, with boundary conditions

$$\phi(0) = 0 \qquad \text{and} \qquad \phi(\pm\infty) = \pm\phi_\infty\ , \tag{18}$$

given implicitly by[7]

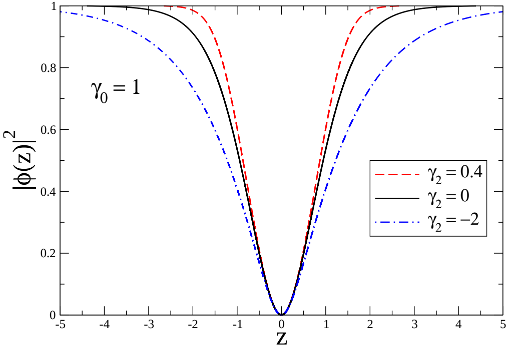
$$\int_0^{\phi(z)^2} \frac{\sqrt{1 - 2\,\gamma_2\,\xi}}{\sqrt{\xi}|\phi_\infty^2 - \xi|}\,d\xi = 2\sqrt{\gamma_0}\,z \tag{19}$$

Setting $\gamma_2 = 0$ one recovers the familiar result

$$\phi(z) = \phi_\infty \tanh\left(\sqrt{\gamma}\,\phi_\infty\,z\right)\ . \tag{20}$$

[7] F. Sgarlata, G. Mazzarella, and LS, J. Phys. B **48**, 115301 (2015).

Solitons with the 1D MGPE (IV)



Dark solitons of the 1D MGPE with $\gamma_0 = 1$ and three values of the finite-range strength γ_2. In the plots we set $\phi_\infty = 1$.

Solitons with the 1D MGPE (V)

If $\gamma_0 < 0$ and $\gamma_2 < 1/2$ the 1D MGPE admits exact bright soliton solutions $\phi(z)$, with boundary conditions

$$\phi(0) = \phi_0 \qquad \text{and} \qquad \phi(\pm\infty) = 0 \,, \tag{21}$$

given implicitly by[8]

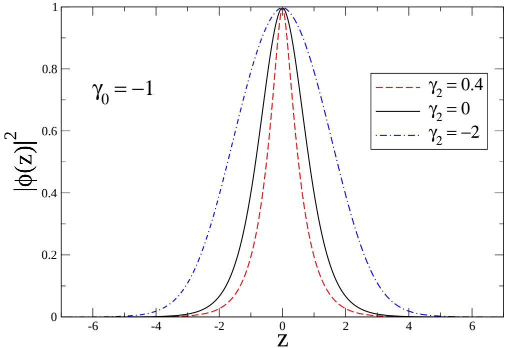
$$\int_{\phi(z)^2}^{\phi_0^2} \frac{\sqrt{1 - 2\,\gamma_2\,\xi}}{\xi\sqrt{\phi_0^2 - \xi}}\, d\xi = 2\,\sqrt{|\gamma_0|}\, z \tag{22}$$

Setting $\gamma_2 = 0$ one recovers the familiar result

$$\phi(z) = \phi_0 \operatorname{sech}\left(\sqrt{|\gamma_0|}\,\phi_0\, z\right) . \tag{23}$$

[8] F. Sgarlata, G. Mazzarella, and LS, J. Phys. B **48**, 115301 (2015).

Solitons with the 1D MGPE (VI)



Bright solitons of the 1D MGPE with $\gamma_0 = -1$ and three values of the finite-range strength γ_2. In the plots we set $\psi_0 = 1$.

Vortices with the 3D MGPE (I)

Vortex solutions of the MGPE can be found in 3D by considering $U(\mathbf{r}) = 0$ and cylindric symmetry, i.e.

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + (N-1)g_0|\psi(\rho,\theta)|^2 + g_2(N-1)\nabla^2|\psi(\rho,\theta)|^2 \right] \psi(\rho,\theta) = \mu\,\psi(\rho,\theta) \,, \tag{24}$$

setting

$$\psi(\rho,\theta) = \phi(\rho)\, e^{ik\theta} \,, \tag{25}$$

with ρ cylindric radial coordinate, θ cylindric angular coordinate, and k quantum number of circulation. Then one finds

$$\left[-\frac{\hbar^2}{2m}\nabla_\rho^2 + \frac{\hbar^2 k^2}{2m\rho^2} + (N-1)g_0\phi(\rho)^2 + (N-1)g_2\nabla_\rho^2\phi(\rho)^2 \right] \phi(\rho) = \mu\,\phi(\rho) \,, \tag{26}$$

where

$$\nabla_\rho^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) \,. \tag{27}$$

Vortices with the 3D MGPE (II)

Imposing the boundary condition

$$\phi(+\infty) = \phi_\infty \qquad \text{which gives} \qquad \mu = \gamma_0 (N-1)\phi_\infty^2 \tag{28}$$

and introducing the adimensional variable

$$x = \frac{\rho}{\xi} \qquad \text{with} \qquad \xi = \sqrt{\frac{\hbar^2}{2\, m\, \mu}} \tag{29}$$

and wavefunction

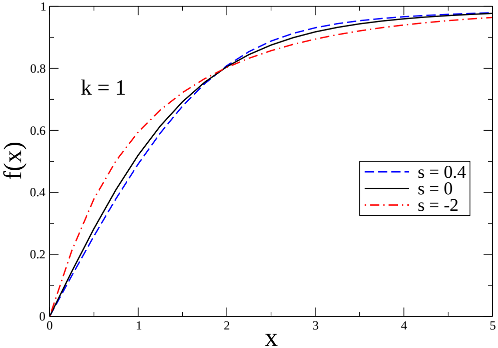
$$f(x) = \frac{\phi(\rho = \xi\, x)}{\phi_\infty}\ , \tag{30}$$

the 1D MGPE becomes

$$\left[-\nabla_x^2 + \frac{k^2}{x^2} + f(x)^2 + r\, \nabla_x^2 f(x)^2 \right] f(x) = f(x) \tag{31}$$

with $s = g_2/(\xi^2 g_0)$.

Vortices with the 3D MGPE (III)



Quantized vortices of the 1D MGPE with $\gamma_0 > 0$ and three values of the finite-range parameter $s = g_2/(\xi^2 g_0)$. In the plots we set the quantum number $k = 1$.

Conclusions

- We have derived a modified Gross-Pitaevskii equation (MGPE) from the Hartree equation
- Our MGPE takes into account finite-range effects
- MGPE admits dark solitons, bright solitons and quantized vortices which generalize the familiar ones of GPE
- Finite-range effects can be measured with Bose-Einstein condensates by increasing the density, i.e. by reducing the average distance between bosons.

Acknowledgements

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