Population imbalance and condensate fraction with SU(3) superfluid fermions

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Summary

- 1. Hamiltonian of the three-component Fermi gas
- 2. Mean-field BCS equations and condensate fraction
- 3. Results of the 3D model
- 4. Results of the 2D model
- 5. Inclusion of a harmonic trap
- Conclusions

1. Hamiltonian of the three-component Fermi gas (I)

The shifted Hamiltonian density of a dilute and interacting three-hyperfine-component Fermi gas in a volume V is given by

$$\begin{aligned}\hat{\mathcal{H}}' &= \sum_{\alpha=R,G,B} \hat{\psi}_\alpha^+ \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\hat{\psi}_\alpha \\ &+ g\left(\hat{\psi}_R^+\hat{\psi}_G^+\hat{\psi}_G\hat{\psi}_R + \hat{\psi}_R^+\hat{\psi}_B^+\hat{\psi}_B\hat{\psi}_R + \hat{\psi}_G^+\hat{\psi}_B^+\hat{\psi}_B\hat{\psi}_G\right) , \end{aligned} \tag{1}$$

where $\hat{\psi}_\alpha(\mathbf{r})$ is the field operator that destroys a fermion of component α in the position $\mathbf{r}$. To mimic QCD the three components are thought as three colors: red (R), green (G) and blue (B).
The attractive inter-atomic interaction is described by a contact pseudo-potential of strength g ($g < 0$).

1. Hamiltonian of the three-component Fermi gas (II)

The number density operator is

$$\hat{n}(\mathbf{r}) = \sum_{\alpha=R,G,B} \hat{\psi}_\alpha^+(\mathbf{r})\hat{\psi}_\alpha(\mathbf{r}) \tag{2}$$

and the average number of fermions reads

$$N = \int d^3\mathbf{r}\, \langle \hat{n}(\mathbf{r}) \rangle \ . \tag{3}$$

This total number N is fixed by the chemical potential μ which appears in Eq. (1).
As stressed by Ozawa and Baym [Phys. Rev. A **82**, 063615 (2010)] by fixing **only** the total chemical potential μ (or equivalently only the total number of atoms N) the Hamiltonian (1) is invariant under global SU(3) rotations of the species.

2. Mean-field BCS equations and condensate fraction (I)

As shown by Ozawa and Baym [Phys. Rev. A **82**, 063615 (2010)], the attractive interaction ($g < 0$) leads to pairing of fermions which breaks the SU(3) symmetry but **only two colors are paired and one is left unpaired**.

We assume that **the red and green particles are paired** and **the blue ones are not paired**. The interacting terms can be then treated within the minimal mean-field BCS approximation, giving

$$g\ \hat{\psi}_R^+\hat{\psi}_G^+\hat{\psi}_G\hat{\psi}_R = g\ \langle\hat{\psi}_R^+\hat{\psi}_G^+\rangle\hat{\psi}_G\hat{\psi}_R + g\ \hat{\psi}_R^+\hat{\psi}_G^+\langle\hat{\psi}_G\hat{\psi}_R\rangle \qquad (4)$$

and

$$g\ \hat{\psi}_R^+\hat{\psi}_B^+\hat{\psi}_B\hat{\psi}_R = g\ \hat{\psi}_G^+\hat{\psi}_B^+\hat{\psi}_B\hat{\psi}_G = 0\ . \qquad (5)$$

Notice that the Hartree terms have been neglected, while the **pairing gap** $\Delta = g\langle\hat{\psi}_G\hat{\psi}_R\rangle$ between red and green fermions is the key quantity.

2. Mean-field BCS equations and condensate fraction (II)

The shifted Hamiltonian density (1) is diagonalized by using the Bogoliubov-Valatin representation of the field operator $\hat{\psi}_\alpha(\mathbf{r})$ in terms of the anticommuting quasi-particle Bogoliubov operators $\hat{b}_{\mathbf{k}\alpha}$ with **quasi-particle amplitudes** u_k and v_k and energy E_k. After minimization of the resulting quadratic Hamiltonian one finds familiar expressions for these quantities:

$$E_k = \left[\left(\frac{\hbar^2 k^2}{2m} - \mu \right)^2 + \Delta^2 \right]^{1/2} \quad (6)$$

and

$$v_k^2 = \frac{1}{2} \left(1 - \frac{\frac{\hbar^2 k^2}{2m} - \mu}{E_k} \right) , \quad (7)$$

with $u_k^2 = 1 - v_k^2$.

2. Mean-field BCS equations and condensate fraction (III)

In addition we find the equation for the number of particles

$$N = N_R + N_G + N_B \ , \tag{8}$$

where

$$N_R = N_G = \frac{1}{2} \sum_{\mathbf{k}} v_k^2 \tag{9}$$

and

$$N_B = \sum_{\mathbf{k}} \Theta \left(\mu - \frac{\hbar^2 k^2}{2m} \right) \ , \tag{10}$$

with $\Theta(x)$ the Heaviside step function, and also the gap equation

$$-\frac{1}{g} = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2E_k} \ . \tag{11}$$

The chemical potential μ and the gap energy Δ are obtained by solving equations (8) and (11).

2. Mean-field BCS equations and condensate fraction (IV)

We observe that the **condensate number of red-green pairs** is given by

$$N_0 = \int d^3\mathbf{r}_1 \; d^3\mathbf{r}_2 \; |\langle \hat{\psi}_G(\mathbf{r}_1) \hat{\psi}_R(\mathbf{r}_2) \rangle|^2 , \qquad (12)$$

and it is straightforward to show that

$$N_0 = \sum_{\mathbf{k}} u_k^2 v_k^2 \; . \qquad (13)$$

For details see: L.S., N. Manini, and A. Parola, Phys. Rev. A **72**, 023621 (2005); G. Ortiz and J. Dukelsky, Phys. Rev. A **72**, 043611 (2005); N. Fukushima, Y. Ohashi, E. Taylor, and A. Griffin, Phys. Rev. A **75**, 033609 (2007).
Due to the choice of a contact potential, **the gap equation (11) diverges in the ultraviolet**. This divergence is linear in three dimensions and logarithmic in two dimensions. Let us face this problem in the next two sections.

3. Results of the 3D model (I)

In three dimensions, a suitable regularization (see Marini, Pistolesi, and Strinati, Eur. Phys. J. B **1**, 151 (1998)] is obtained by introducing the **inter-atomic scattering length** a_F via the equation

$$-\frac{1}{g} = -\frac{m}{4\pi\hbar^2 a_F} + \frac{1}{V}\sum_{\mathbf{k}} \frac{m}{\hbar^2 k^2} \,, \tag{14}$$

and then subtracting this equation from the gap equation (11). In this way one obtains the three-dimensional **regularized gap equation**

$$-\frac{m}{4\pi\hbar^2 a_F} = \frac{1}{V}\sum_{\mathbf{k}} \left(\frac{1}{2E_k} - \frac{m}{\hbar^2 k^2} \right) . \tag{15}$$

3. Results of the 3D model (II)

In the three-dimensional continuum limit $\sum_{\mathbf{k}} \to V/(2\pi^2) \int k^2 dk$ from the number equation (8) with (9) and (10) we find the total number density as

$$n = \frac{N}{V} = n_R + n_G + n_B \; , \tag{16}$$

with

$$n_R = n_G = \frac{1}{2}\frac{(2m)^{3/2}}{2\pi^2\hbar^3}\,\Delta^{3/2}\, I_2\left(\frac{\mu}{\Delta}\right) \; , \tag{17}$$

and

$$n_B = \frac{1}{3}\frac{(2m)^{3/2}}{2\pi^2\hbar^3}\mu^{3/2}\;\Theta(\mu) \; . \tag{18}$$

The renormalized gap equation (15) becomes instead

$$-\frac{1}{a_F} = \frac{2(2m)^{1/2}}{\pi\hbar^3}\,\Delta^{1/2}\, I_1\left(\frac{\mu}{\Delta}\right) \; , \tag{19}$$

where $k_F = (6\pi N/(3V))^{1/3} = (2\pi^2 n)^{1/3}$ is the Fermi wave number.

3. Results of the 3D model (III)

Here $I_1(x)$ and $I_2(x)$ are the two monotonic functions

$$I_1(x) = \int_0^{+\infty} y^2 \left(\frac{1}{\sqrt{(y^2 - x)^2 + 1}} - \frac{1}{y^2} \right) dy \ , \tag{20}$$

$$I_2(x) = \int_0^{+\infty} y^2 \left(1 - \frac{y^2 - x}{\sqrt{(y^2 - x)^2 + 1}} \right) dy \ , \tag{21}$$

which can be expressed in terms of elliptic integrals, as shown by Marini, Pistolesi and Strinati [Eur. Phys. J. B **1**, 151 (1998)].
In a similar way we get the **condensate density of the red-green pairs** as

$$n_0 = \frac{N_0}{V} = \frac{m^{3/2}}{8\pi\hbar^3} \Delta^{3/2} \sqrt{\frac{\mu}{\Delta} + \sqrt{1 + \frac{\mu^2}{\Delta^2}}} \ . \tag{22}$$

3. Results of the 3D model (IV)

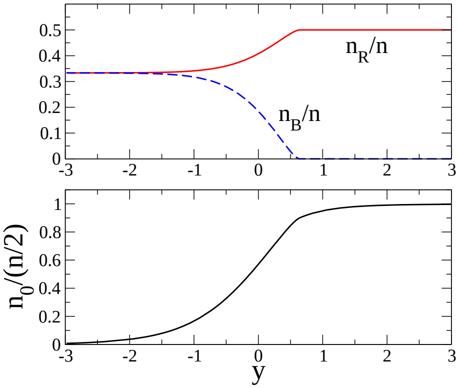

Figure: Upper panel: fraction of red fermions n_R/n (solid line) and fraction of blue fermions n_B/n (dashed line) as a function of scaled interaction strength $y = 1/(k_F a_F)$. Lower panel: **condensed fraction of red-green particles** n_0/n as a function of scaled interaction strength $y = 1/(k_F a_F)$. Note that $n_R/n = n_G/n$.

4. Results of the 2D model (I)

Contrary to the three-dimensional case, in two dimensions quite generally a **bound-state energy** ϵ_B exists for any value of the interaction strength g between atoms. For the contact potential the bound-state equation is

$$-\frac{1}{g} = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 k^2}{2m} + \epsilon_B} , \tag{23}$$

and then subtracting this equation from the gap equation (11) one obtains the **two-dimensional regularized gap equation** (see Marini, Pistolesi, and Strinati, Eur. Phys. J. B **1**, 151 (1998)]

$$\sum_{\mathbf{k}} \left(\frac{1}{\frac{\hbar^2 k^2}{2m} + \epsilon_B} - \frac{1}{2E_k} \right) = 0 . \tag{24}$$

4. Results of the 2D model (II)

In the two-dimensional continuum limit $\sum_{\mathbf{k}} \to V/(2\pi) \int kdk$, the regularized gap equation gives

$$\epsilon_B = \Delta \left(\sqrt{1 + \frac{\mu^2}{\Delta^2}} - \frac{\mu}{\Delta} \right) . \quad (25)$$

Instead, from the number equation we get

$$n = \frac{N}{V} = n_R + n_G + n_B , \quad (26)$$

where V is a two-dimensional volume (i.e. an area), and

$$n_R = n_G = \frac{1}{2}\left(\frac{m}{2\pi\hbar^2}\right)\Delta \left(\frac{\mu}{\Delta} + \sqrt{1 + \frac{\mu^2}{\Delta^2}} \right) , \quad (27)$$

$$n_B = \left(\frac{m}{2\pi\hbar^2}\right)\mu \, \Theta(\mu) . \quad (28)$$

Finally, the **condensate density of red-green pairs** is given by

$$n_0 = \frac{1}{4}\left(\frac{m}{2\pi\hbar^2}\right)\Delta \left(\frac{\pi}{2} + \arctan\left(\frac{\mu}{\Delta}\right) \right) . \quad (29)$$

4. Results of the 2D model (III)

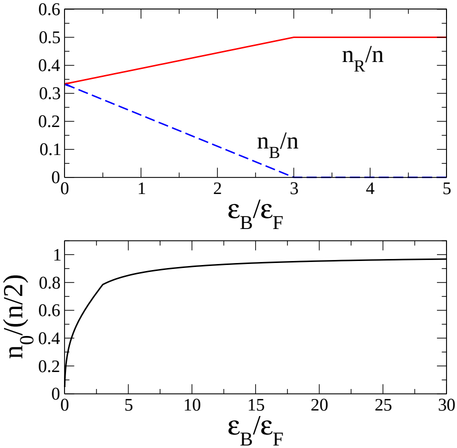

Figure: Upper panel: fraction of red fermions n_R/n (solid line) and fraction of blue fermions n_B/n (dashed line) as a function of scaled bound-state energy ϵ_B/ϵ_F. Lower panel: **condensed fraction of red-green particles** n_0/n as a function of scaled bound-state energy ϵ_B/ϵ_F. Note that $n_R/n = n_G/n$.

5. Inclusion of a harmonic trap (I)

It is interesting to study the effect of a harmonic potential

$$U(r) = \frac{1}{2} m\omega^2 r^2 \qquad (30)$$

on the properties of the three-component ultracold gas in the BCS-BEC crossover. **For semplicity** we investigate **the two-dimensional case**, which gives rise to elegant formulas also in this non-uniform configuration.

In fact, by using the local density approximation, namely the substitution

$$\mu \to \mu(r) = \bar{\mu} - U(r) \ , \qquad (31)$$

the gap equation (25) gives the space-dependent gap parameter as

$$\Delta(r) = \Delta_0 \ (1 - \frac{r^2}{r_0^2}) \ \Theta(1 - \frac{r^2}{r_0^2}) \ , \qquad (32)$$

where $\Delta_0 = \sqrt{\epsilon_B^2 + 2\epsilon_B \bar{\mu}}$ and $r_0 = \Delta_0 / \sqrt{\epsilon_B m\omega}$. Here $\bar{\mu}$ is the chemical potential of the non-uniform system.

5. Inclusion of a harmonic trap (II)

In the same way the **density profiles** of red, green and blue fermions read

$$n_R(r) = n_G(r) = \frac{1}{2}(\frac{m}{2\pi\hbar^2})\Delta(r)\left(\frac{\mu(r)}{\Delta(r)} + \sqrt{1 + \frac{\mu(r)^2}{\Delta(r)^2}}\right) , \qquad (33)$$

$$n_B(r) = (\frac{m}{2\pi\hbar^2})\mu(r)\ \Theta(\mu(r)) . \qquad (34)$$

The **density profile of condensed red-green pairs** is instead given by

$$n_0(r) = \frac{1}{4}(\frac{m}{2\pi\hbar^2})\Delta(r)\left(\frac{\pi}{2} + \arctan(\frac{\mu(r)}{\Delta(r)})\right) . \qquad (35)$$

5. Inclusion of a harmonic trap (III)

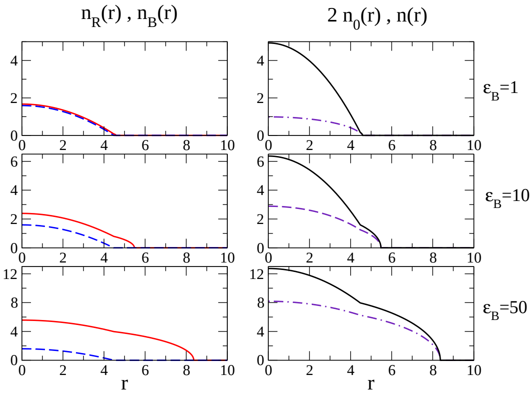

Figure: Left panels: density profile $n_R(r)$ of red fermions (solid lines) and density profile $n_B(r)$ of blue fermions (dashed lines). Right panels: total density profile $n(r) = 2n_R(r) + n_B(r)$ (solid lines) and **condensate density profile** $2n_0(r)$ (dot-dashed lines). Results obtained with $\bar{\mu} = 10$ and three values of the bound-state energy ϵ_B. Note that $n_R(r) = n_G(r)$.

Conclusions

- We have investigated the condensate fraction and the population imbalance of a three-component ultracold fermions by increasing the SU(3) invariant attractive interaction
- We have considered the superfluid system both in the three-dimensional case and in the two-dimensional one.
- We have obtained **explicit formulas** and plots for number densities, condensate density and population imbalance in the full BCS-BEC crossover.
- Our results can be of interest for next future experiments with degenerate gases made of alkali-metal or alkaline-earth atoms in three hyperfine states.
- The problem of unequal couplings, and also that of a fixed number of atoms for each component, with the inclusion of more than one order parameter, is under investigation.