# Density functional and hydrodynamics equations for the confined unitary Fermi gas

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# Summary

- Extended Thomas-Fermi density functional
- Finding the universal parameters of the ETF functional
- Odd-even splitting
- Extended superfluid hydrodynamics
- Sound velocity and collective modes
- Conclusions

# Extended Thomas-Fermi density functional

The Thomas-Fermi (TF) energy functional* of a dilute and ultracold two-component Fermi gas trapped by an external potential $U(\mathbf{r})$ is

$$E_{TF} = \int d^3\mathbf{r}\ n(\mathbf{r})[\varepsilon(n(\mathbf{r}); a_F) + U(\mathbf{r})]\ , \tag{1}$$

with $\varepsilon(n; a_F)$ bulk energy per particle, $n(\mathbf{r})$ total density and $a_F$ the s-wave scattering length. The total number of fermions is

$$N = \int d^3\mathbf{r}\ n(\mathbf{r})\ . \tag{2}$$

By minimizing $E_{TF}$ one finds

$$\mu(n(\mathbf{r}); a_F) + U(\mathbf{r}) = \bar{\mu}\ , \tag{3}$$

with $\mu(n; a_F) = \frac{\partial(n\varepsilon(n;a_F))}{\partial n}$ bulk chemical potential of a uniform system and $\bar{\mu}$ chemical potential of the non uniform system.

*S. Giorgini, L.P. Pitaevskii, and S. Stringari, RMP **80**, 1215 (2008).

For the uniform unitary Fermi gas[†] the s-wave scattering length $a_F$ diverges:

$$a_F \to \pm\infty \,, \tag{4}$$

and the only length characterizing the uniform system is the average distance between particles $n^{-1/3}$. In this case:

$$\varepsilon(n;\xi) = \xi \frac{3}{5}\frac{\hbar^2}{2m}(3\pi^2)^{2/3} n^{2/3} = \xi \, \frac{3}{5} \, \epsilon_F \,, \tag{5}$$

with $\epsilon_F$ Fermi energy of the ideal gas and $\xi$ a universal parameter.

The bulk chemical potential associated to Eq. (5) is

$$\mu(n;\xi) = \frac{\partial(n\varepsilon(n))}{\partial n} = \xi \frac{\hbar^2}{2m}(3\pi^2)^{2/3} n^{2/3} = \xi \, \epsilon_F \,. \tag{6}$$

[†] "The Many-Body X Challenge Problem", formulated by G.F. Bertsch, see R. A. Bishop, IJMP B **15**, iii (2001).

The TF functional must be extended to cure the pathological TF behavior at the surface.

We add to the energy per particle the term

$$\lambda \frac{\hbar^2}{8m} \frac{(\nabla n)^2}{n^2} = \lambda \frac{\hbar^2}{2m} \frac{(\nabla \sqrt{n})^2}{n} \, . \tag{7}$$

Historically, this term was introduced by von Weizsäcker[‡] to treat surface effects in nuclei. Here we consider $\lambda$ as a phenomenological parameter accounting for the increase of kinetic energy due the spatial variation of the density.

Other recent density-functional methods for unitary Fermi gas:
– the Kohn-Sham density functional approach of Papenbrock,
PRA **72**, 041603 (2005);
– the superfluid local-density approximation (SLDA) of Bulgac,
PRA **76**, 040502(R) (2007).

[‡]C.F. von Weizsäcker, ZP **96**, 431 (1935).

The new energy functional, that is the extended Thomas-Fermi (ETF) functional of the unitary Fermi gas, reads

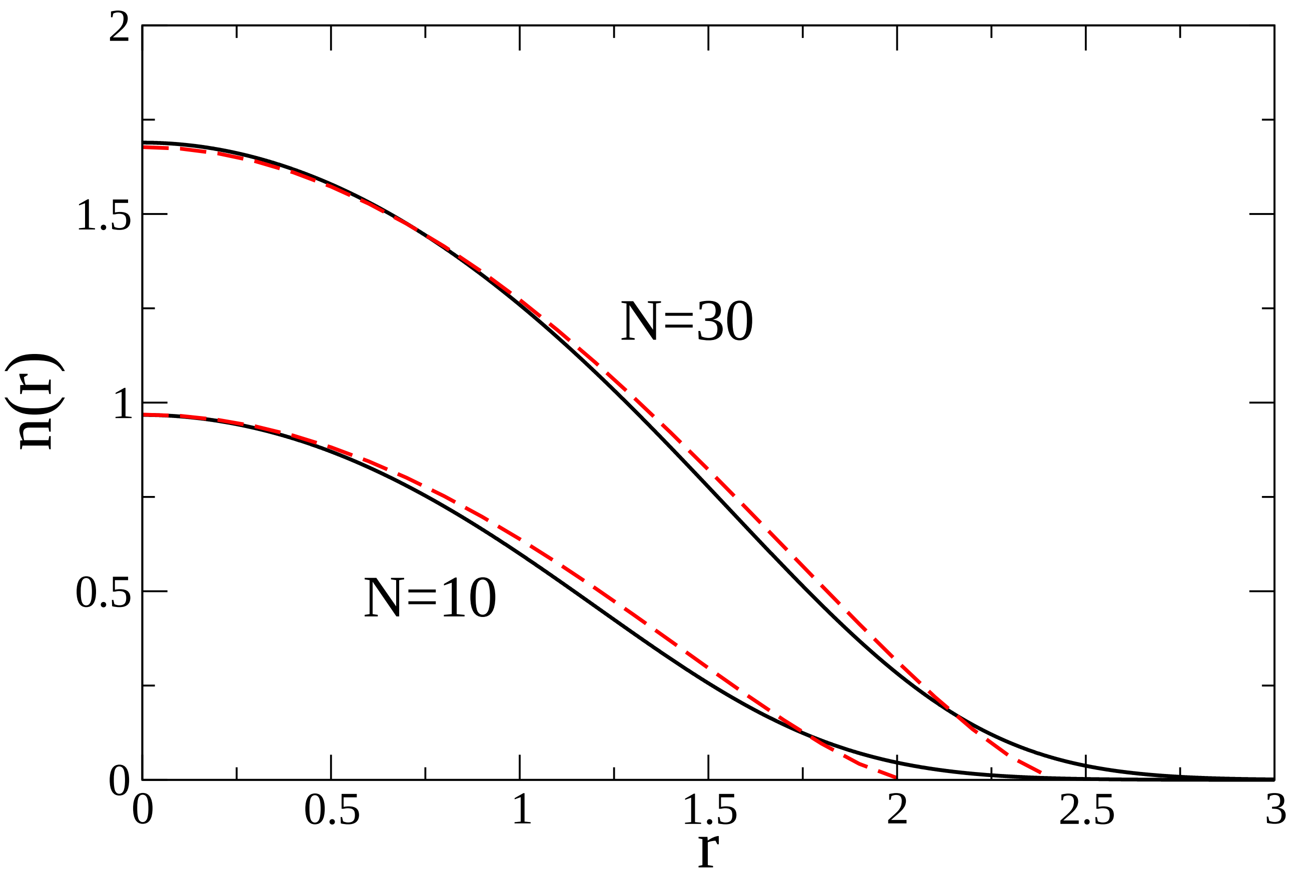
$$E = \int d^3\mathbf{r}\ n(\mathbf{r}) \left[ \lambda \frac{\hbar^2}{8m} \frac{(\nabla n(\mathbf{r}))^2}{n(\mathbf{r})^2} + \xi \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n(\mathbf{r})^{2/3} + U(\mathbf{r}) \right] \quad . \tag{8}$$

By minimizing the ETF energy functional one gets:

$$\left[ -\lambda \frac{\hbar^2}{2m} \nabla^2 + \xi \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n(\mathbf{r})^{2/3} + U(\mathbf{r}) \right] \sqrt{n(\mathbf{r})} = \bar{\mu}\ \sqrt{n(\mathbf{r})} \, . \tag{9}$$

This is a sort of stationary 3D nonlinear Schrödinger (3D NLS) equation.

The constants $\xi$ and $\lambda$ should be universal i.e. independent on the confining potential $U(\mathbf{r})$.




Unitary Fermi gas under harmonic confinement of frequency $\omega$. Density profiles $n(r)$ for $N = 10$ and $N = 30$ fermions obtained with ETF (solid lines) and TF (dashed lines). In all calculations: universal parameter $\xi = 0.44$ and gradient coefficient $\lambda = 1/4$. Lengths in units of $a_H = \sqrt{\hbar/(m\omega)}$. [L.S. and F. Toigo, PRA **78**, 053626 (2008)]

In recent papers [S.K. Adhikari and L.S., PRA **78**, 043616 (2008); L.S. and F. Toigo, PRA **78**, 053626 (2008); S.K. Adhikari and L.S., NJP **11**, 023011 (2009)] we have used this simple choice

$$\xi = 0.44 \qquad \text{and} \qquad \lambda = 1/4 \, , \tag{10}$$

for the unitary Fermi gas in a spherical harmonic potential

$$U(\mathbf{r}) = \frac{1}{2} m \omega^2 r^2 \, . \tag{11}$$

$\xi = 0.44$ is a MC prediction for uniform unitary Fermi gas [Carlson et al. PRL **91** 050401 (2003)].

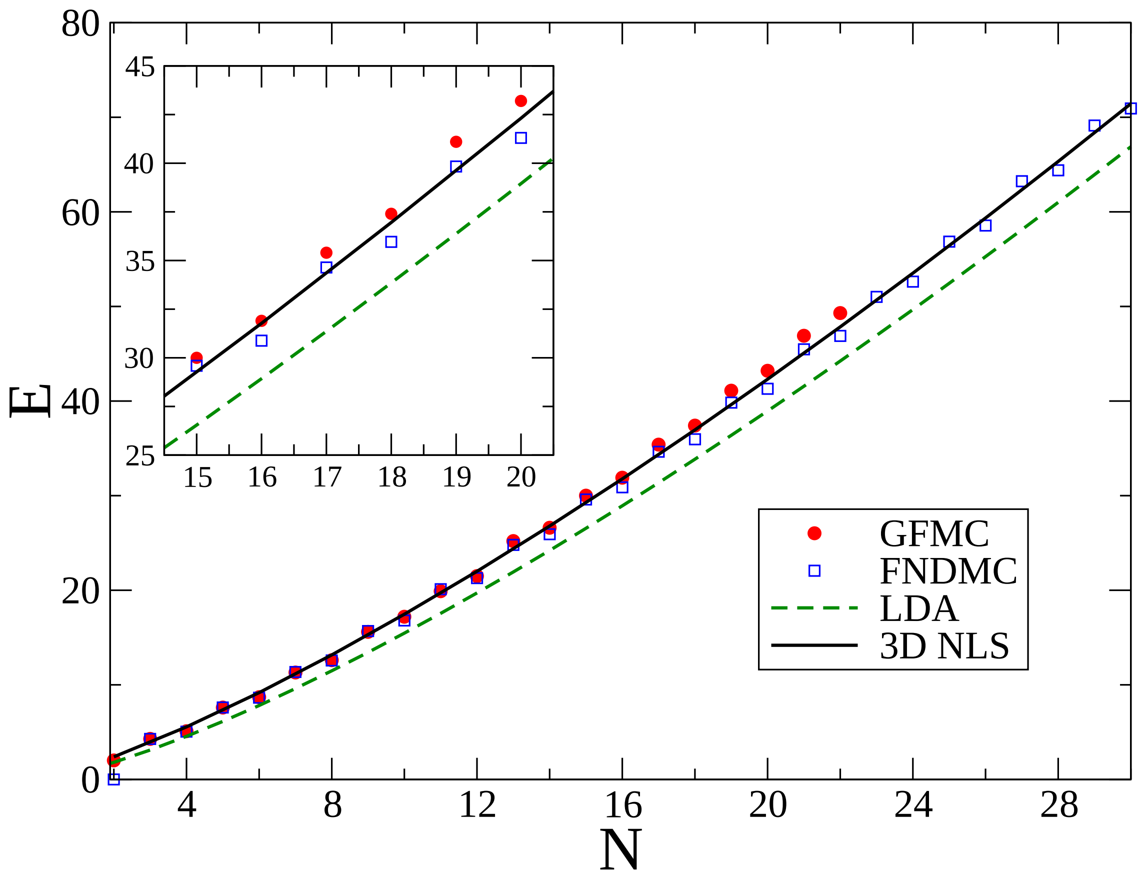
$\lambda \simeq 0.25$ is the prediction at unitarity of effective field theory [G. Rupak and T. Schäfer, NP A **816**, 52 (2009)].

Here we compare the results of our 3D NLS equation with recent Monte Carlo data§:

– Green-function Monte Carlo (GFMC) of Chang and Bertsch, PRA **76** 021603(R) (2007);

– Fixed-node diffusion Monte Carlo (FNDMC) of Dörte Blume *et al.* PRL **99**, 233201 (2007).

§They compare their MC data with an approximate $N$-expansion using $\lambda = \xi/9 \simeq 0.05$.




Ground-state energy $E$ (in units of $\hbar\omega$) versus $N$. Solid line: ETF functional with $\xi = 0.44$ and $\lambda = 1/4$. Dashed line: local density approximation (LDA), i.e. the Thomas-Fermi model ($\lambda = 0$). The results of Green-function Monte Carlo (GFMC) and fixed-node diffusion Monte Carlo (FNDMC) are shown for a comparison (symbols). [S.K. Adhikari and L.S., NJP **11**, 023011 (2009)]

# Finding the universal parameters of the ETF functional

To determine $\xi$ and $\lambda$ we look for the values of the two parameters which lead to the best fit of the ground-state energies obtained by Monte Carlo data.

We use the **more recent and reliable** Monte Carlo results with $N$ even (complete superfluidity): the fixed-node diffusion Monte Carlo (FNDMC) of J von Stecher, C.H. Greene and D. Blume, PRA **77** 043619 (2008).

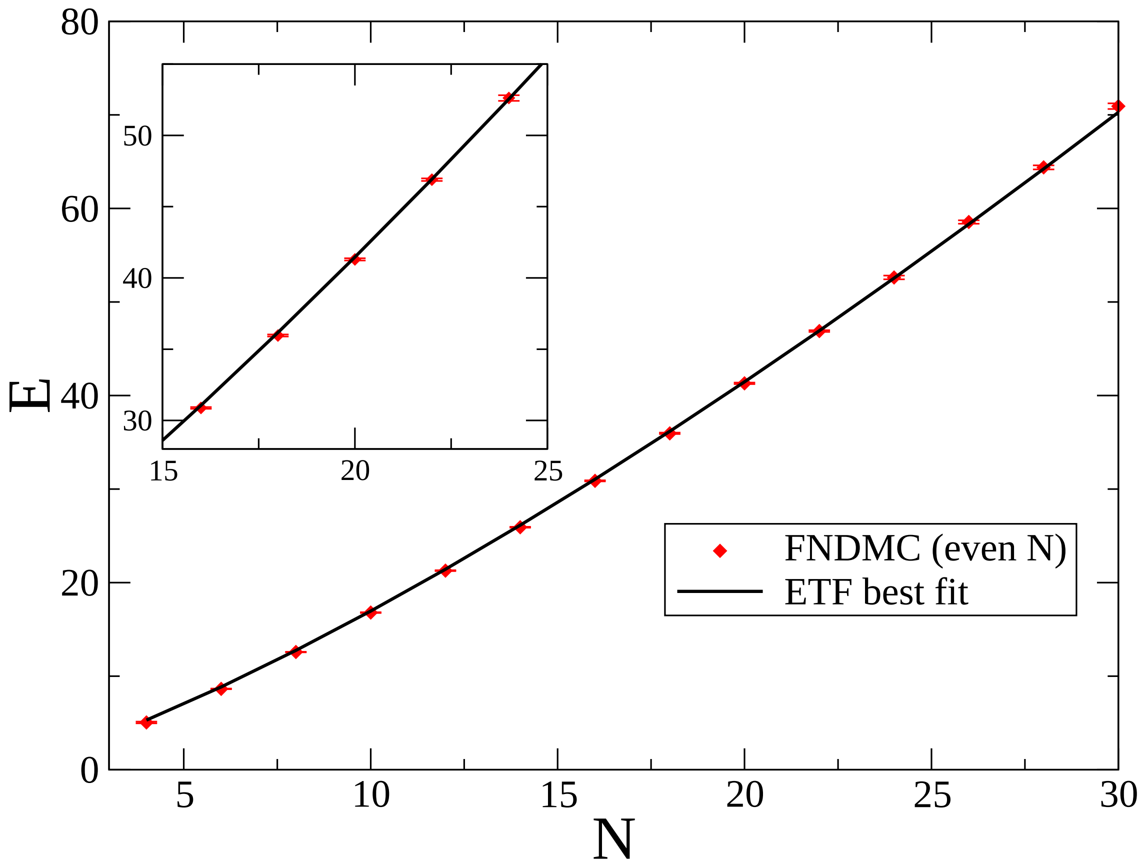
After a systematic analysis [L.S. and F. Toigo, PRA **78**, 053626 (2008)] we find

$$\xi = 0.455 \qquad \text{and} \qquad \lambda = 0.13$$

as the best fitting parameters in the unitary regime.[¶] See the next figure.

Fixing $\xi = 0.44$ we find instead $\lambda = 0.18$.

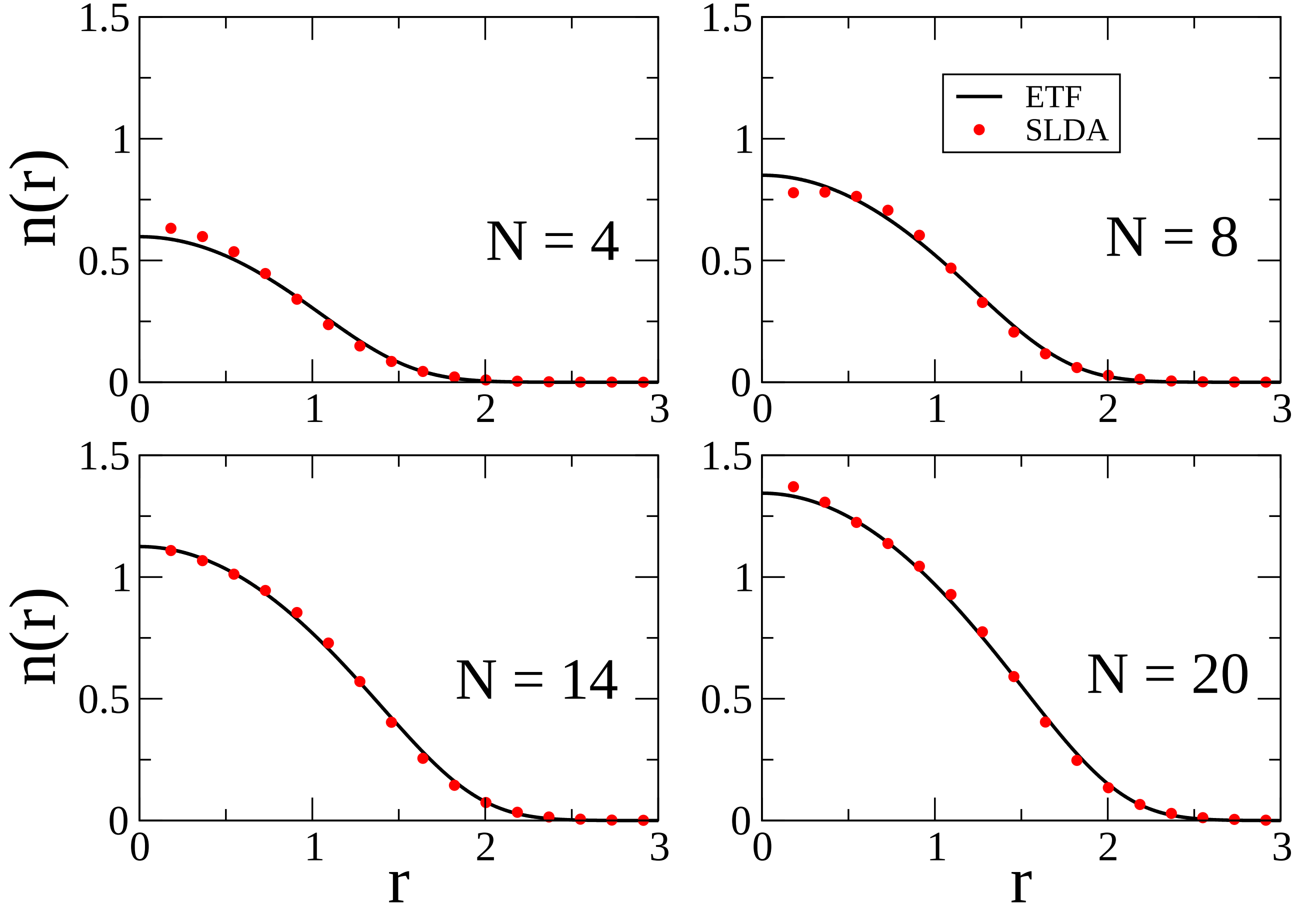
[¶] The value $\xi = 0.455$ coincides with that obtained by A. Perali, P. Pieri, and G.C. Strinati, PRL **93**, 100404 (2004) with beyond-mean-field extended BCS theory.




Ground-state energy $E$ for the unitary Fermi gas of $N$ atoms under harmonic confinement of frequency $\omega$. Symbols: FNDMC data with even $N$; solid line: ETF results with best fit ($\xi = 0.455$ and $\lambda = 0.13$). Energy in units of $\hbar\omega$. [L.S. and F. Toigo, PRA **78**, 053626 (2008)]

| N | $E_{FNDMC}$ | $E_{ETF}$ | $E_{SLDA}$ |
|---|---|---|---|
| 2 | 2.002 | 2.17 | 2.33 |
| 4 | 5.051 | 5.19 | 5.52 |
| 6 | 8.64 | 8.71 | 9.07 |
| 8 | 12.58 | 12.61 | 12.94 |
| 10 | 16.80 | 16.83 | 17.15 |
| 12 | 21.28 | 21.32 | 21.63 |
| 14 | 25.92 | 26.04 | 26.32 |
| 16 | 30.88 | 30.99 | 31.21 |
| 18 | 35.96 | 36.13 | 36.27 |
| 20 | 41.30 | 41.46 | 41.51 |
| 22 | 46.89 | 46.96 | 46.92 |
| 24 | 52.62 | 52.63 | |
| 26 | 58.55 | 58.45 | |
| 28 | 64.39 | 64.41 | |
| 30 | 70.93 | 70.51 | |

Table 1. Ground-state energy $E$ of the unitary Fermi gas of $N$ even atoms under harmonic confinement, in units of the harmonic energy $\hbar\omega$. Comparison among different calculations: fixed node diffusion Monte Carlo ($E_{FNDMC}$), our optimized extended Thomas-Fermi with $\xi = 0.455$ and $\lambda = 0.13$ ($E_{ETF}$), and the Bulgac's SLDA ($E_{SLDA}$).




Unitary Fermi gas under harmonic confinement of frequency $\omega$. Density profiles $n(r)$ for $N$ (even) fermions obtained with our ETF (solid lines) and Bulgac's SLDA (circles). Lengths in units of $a_H = \sqrt{\hbar/(m\omega)}$.

# Odd-even splitting

In our determination of $\xi$ and $\lambda$ we have analyzed the unitary gas with an even number $N$ of particles.

Monte Carlo calculations show a clear odd-even effect (zig-zag effect): the ground state energy of $N$ odd particles in the isotropic harmonic trap is

$$E_N = \frac{1}{2}(E_{N-1} + E_{N+1}) + \Delta_N \; , \tag{12}$$

where the splitting $\Delta_N$ is always positive.

Dam Thanh Son has suggested[‖] that, given the superfluid cloud of even particles, the extra particle is localized where the energy gap is smallest, which is near the edge of the cloud.

[‖]D.T. Son, e-preprint arXiv:0707.1851.

Dam Thanh Son has also found that, for fermions at unitarity, confined by a harmonic potential with frequency $\omega$, the odd-even splitting grows as

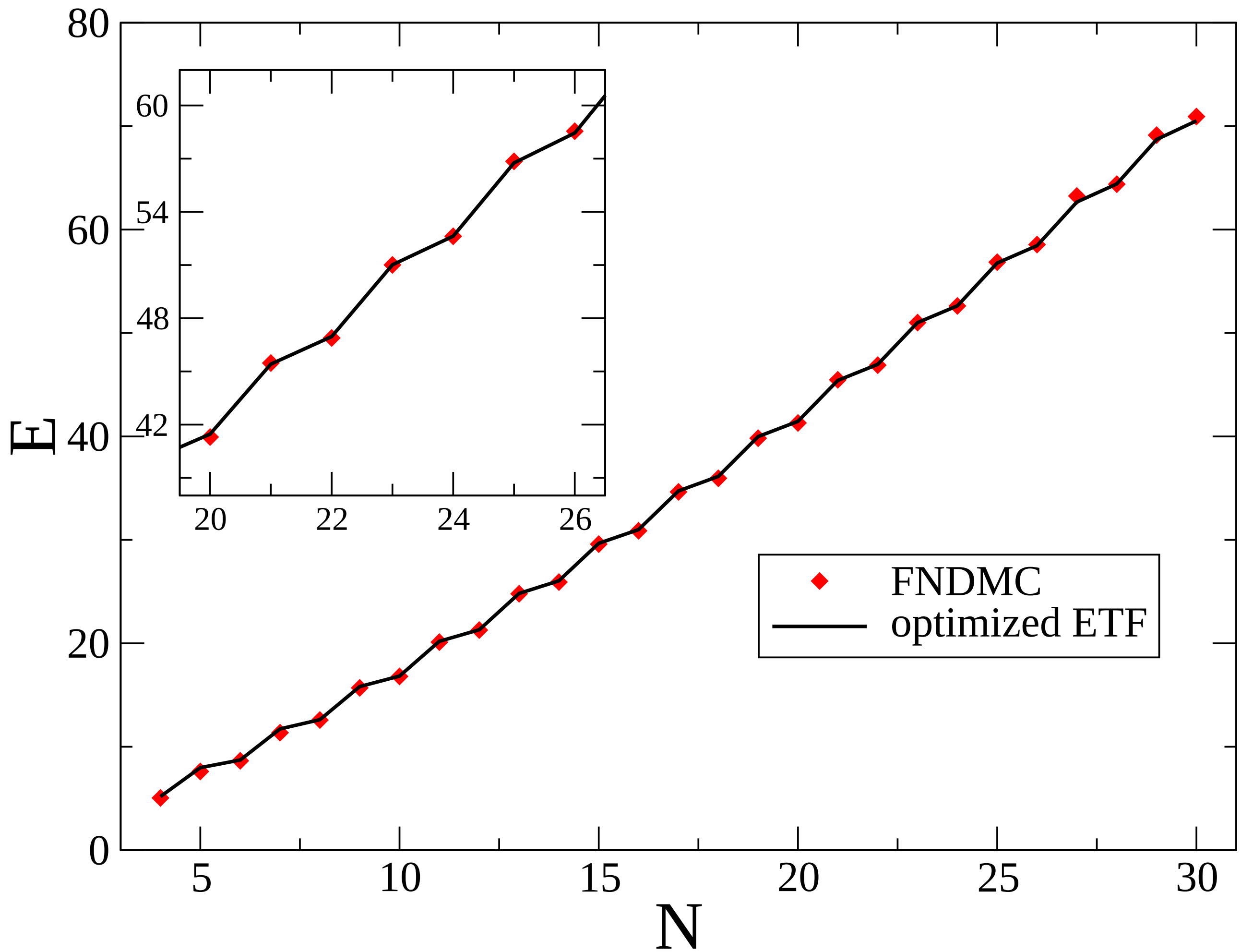
$$\Delta E_N = \gamma \; N^{1/9} \; \hbar\omega \; , \tag{13}$$

where $\gamma$ is an unknown dimensionless constant.

After a systematic investigation of the FNDMC data we find that

$$\gamma = 0.856$$

gives the best fit. See the next figure.




Ground-state energy $E$ for the unitary Fermi gas of $N$ atoms under harmonic confinement of frequency $\omega$. Diamonds: FNDMC data with both even and odd $N$; solid line: optimized ETF results ($\xi = 0.455$, $\lambda = 0.13$, $\gamma = 0.856$). Energy in units of $\hbar\omega$. [L.S. and F. Toigo, PRA **78**, 053626 (2008)]

# Extended superfluid hydrodynamics

Let us now analyze the effect of the gradient term on the dynamics of the superfluid unitary Fermi gas.

At zero temperature the low-energy collective dynamics of this fermionic gas can be described by the equations of extended** irrotational and inviscid hydrodynamics:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 \, , \tag{14}$$

$$m\frac{\partial}{\partial t}\mathbf{v} + \nabla[-\lambda\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} + \mu(n;\xi) + U(\mathbf{r}) + \frac{m}{2}v^2] = 0 \, . \tag{15}$$

They are the simplest extension of the equations of superfluid hydrodynamics of fermions††, where $\lambda = 0$.

**Quantum hydrodynamics of electrons: N. H. March and M. P. Tosi, Proc. R. Soc. A **330**, 373 (1972); E. Zaremba and H.C. Tso, PRB **49**, 8147 (1994).

††S. Giorgini, L.P. Pitaevskii, and S. Stringari, RMP **80**, 1215 (2008).

The extended hydrodynamics equations can be written in terms of a superfluid nonlinear Schrödinger equation (NLSE), which is Galilei-invariant.[‡‡]

In fact, by introducing the complex wave function

$$\psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\; e^{i\theta(\mathbf{r},t)} \;, \tag{16}$$

which is normalized to the total number $N$ of superfluid atoms, and taking into account the correct phase-velocity relationship

$$\mathbf{v}(\mathbf{r},t) = \frac{\hbar}{2m}\nabla\theta(\mathbf{r},t) \;, \tag{17}$$

where $\theta(\mathbf{r},t)$ is the phase of the condensate wavefunction of Cooper pairs, the equation

$$i\hbar\frac{\partial}{\partial t}\psi = [-\frac{\hbar^2}{4m}\nabla^2 + 2U(\mathbf{r}) + 2\mu(|\psi|^2;\xi) + (1-4\lambda)\frac{\hbar^2}{4m}\frac{\nabla^2|\psi|}{|\psi|}]\psi \;, \tag{18}$$

is strictly equivalent to the equations of extended hydrodynamics.

[‡‡]H.-D. Doebner and G.A. Goldin, PRA **54**, 3764 (1996).

The extended hydrodynamics equations are the Euler-Lagrange equation of the following Lagrangian density

$$\mathcal{L} = -n\left(\dot{\theta} + \frac{\hbar^2}{8m}(\nabla\theta)^2 + U(\mathbf{r}) + \varepsilon(n;\xi) + \lambda\frac{\hbar^2}{8m}\frac{(\nabla n)^2}{n^2}\right) , \quad (19)$$

which depends on the total number density $n(\mathbf{r},t)$ and the phase $\theta(\mathbf{r},t)$ [L.S. and F. Toigo, PRA **78**, 053626 (2008)].

In the case $\lambda = 0$ it is called the Popov Lagrangian of superfluid hydrodynamics [V.N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983)].

Setting, as previously,

$$\psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\; e^{i\theta(\mathbf{r},t)} , \quad (20)$$

the extended Popov Lagrangian (19) is equivalent to the following one:

$$\mathcal{L} = \psi^*\left(i\frac{\hbar}{2}\frac{\partial}{\partial t} + \frac{\hbar^2}{8m}\nabla^2 - U(\mathbf{r}) - \varepsilon(|\psi|^2;\xi)\right)\psi + (1-4\lambda)\frac{\hbar^2}{8m}(\nabla|\psi|)^2 . \quad (21)$$

# Sound velocity and collective modes

From the equations of superfluid hydrodynamics one finds the dispersion relation of low-energy collective modes of the uniform ($U(\mathbf{r}) = 0$) unitary Fermi gas in the form

$$\frac{\Omega}{q} = \sqrt{\frac{\xi}{3}} v_F \; , \qquad (22)$$

where $\Omega$ is the collective frequency, $q$ is the wave number and

$$v_F = \sqrt{\frac{2\epsilon_F}{m}} \qquad (23)$$

is the Fermi velocity of a noninteracting Fermi gas.

The equations of extended superfluid hydrodynamics (or the superfluid NLSE) give [L.S. and F. Toigo, PRA **78**, 053626 (2008)] also a correcting term, i.e.

$$\frac{\Omega}{q} = \sqrt{\frac{\xi}{3}} v_F \sqrt{1 + \frac{3\lambda}{\xi} (\frac{\hbar q}{2 m v_F})^2} \; , \qquad (24)$$

which depends on the ratio $\lambda/\xi$.

In the case of harmonic confinement

$$U(\mathbf{r}) = \frac{1}{2} m \omega^2 r^2 \tag{25}$$

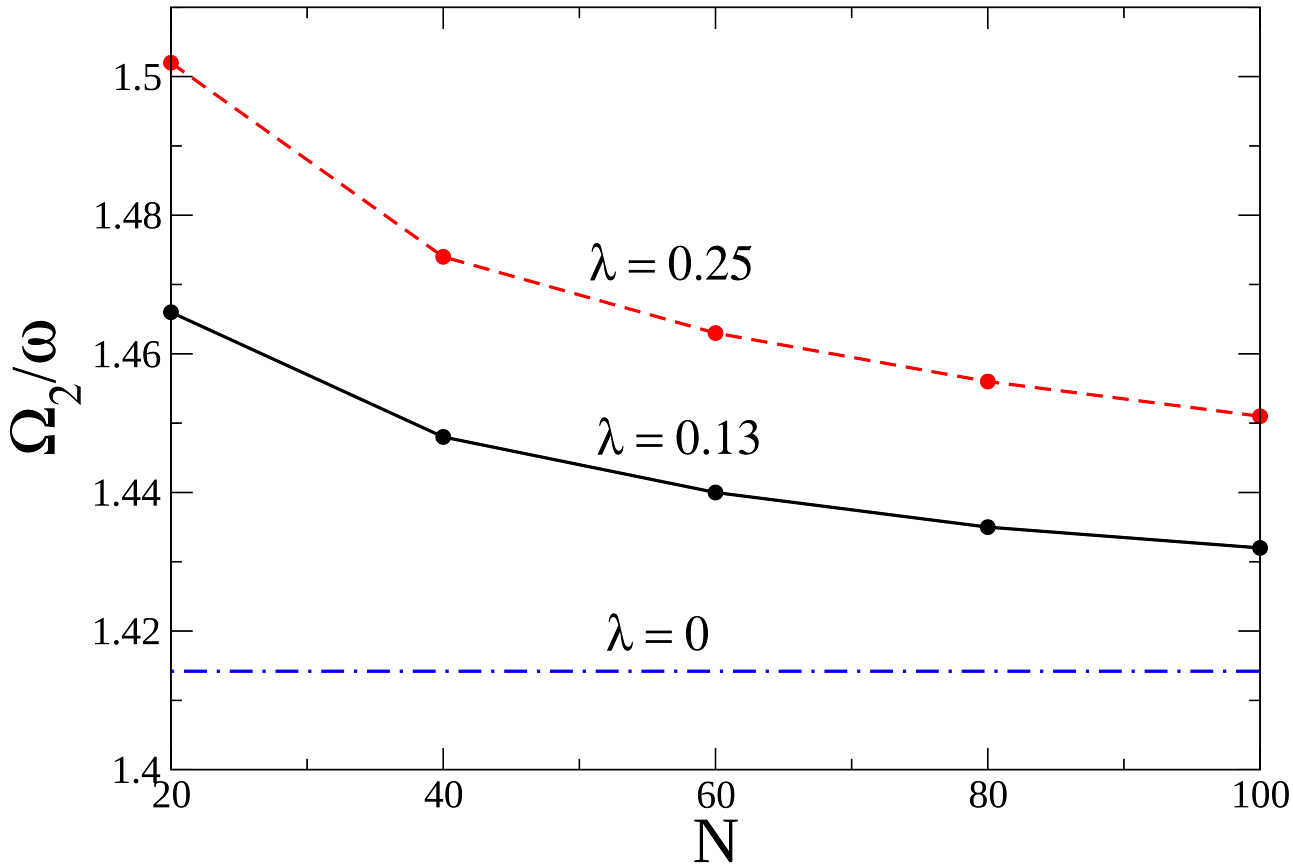
we study numerically the collective modes of the unitary Fermi gas by increasing the number $N$ of atoms.

By solving the superfluid NLSE we find that the frequency $\Omega_0$ of the monopole mode ($l = 0$) and the frequency $\Omega_1$ dipole mode ($l = 1$) do not depend on $N$:

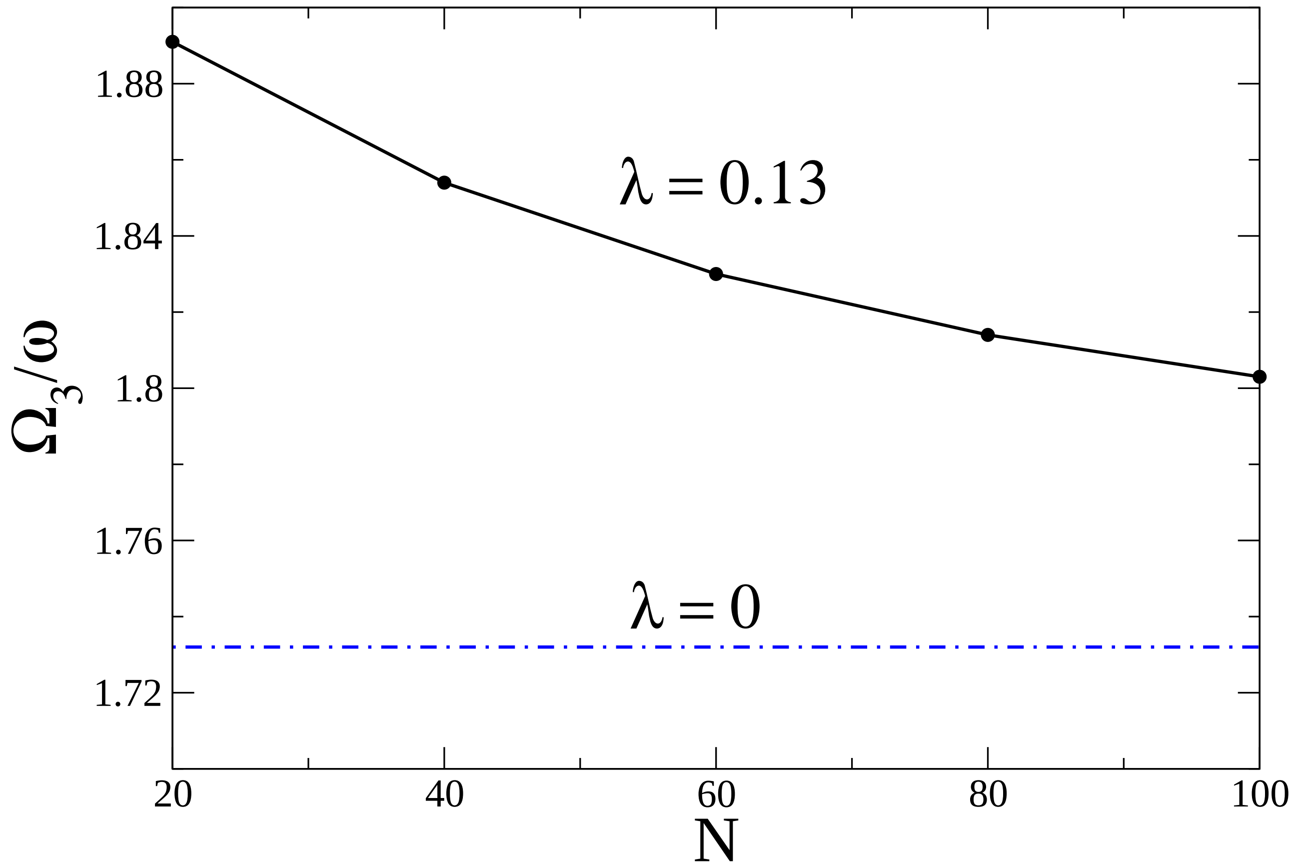
$$\Omega_0 = 2\omega \qquad \text{and} \qquad \Omega_1 = \omega \, , \tag{26}$$

as predicted by Y. Castin [Comptes Rendus Physique **5**, 407 (2004)].

We find instead that the frequencies $\Omega_2$ and $\Omega_3$ of quadrupole ($l = 2$) and octupole ($l = 3$) modes depend on $N$ and on the choice of the gradient coefficient $\lambda$.




Quadrupole frequency $\Omega_2$ of the unitary Fermi gas ($\xi = 0.455$) with $N$ atoms under harmonic confinement of frequency $\omega$. Three different values of the gradient coefficient $\lambda$. For $\lambda = 0$ (TF limit): $\Omega_2 = \sqrt{2}\omega$. [L.S., F. Ancilotto and F. Toigo, preliminary results]




Octupole frequency $\Omega_3$ of the unitary Fermi gas ($\xi = 0.455$) with $N$ atoms under harmonic confinement of frequency $\omega$. Two different values of the gradient coefficient $\lambda$. For $\lambda = 0$ (TF limit): $\Omega_3 = \sqrt{3}\omega$. [L.S., F. Ancilotto and F. Toigo, preliminary results]

# Conclusions

- We have introduced an extended Thomas-Fermi (ETF) functional for the trapped unitary Fermi gas.

- By fitting FNDMC calculations we have determined the universal parameters $\xi$ and $\lambda$ of ETF functional.

- ETF functional can be used to study ground-state density profiles in a generic external potential $U(\mathbf{r})$.

- We have also introduced a time-dependent version of the ETF functional: the extended superfluid hydrodynamics (or superfluid NLSE).

- The superfluid NLSE can be used to investigate collective modes also for a small number of atoms.