Fermionic condensation in ultracold atoms, nuclear matter and neutron stars

Luca Salasnich

Dipartimento di Fisica e Astronomia "Galileo Galilei", Università di Padova, Italy

Prague, July 16, 2013

Collaboration with:
Nicola Manini (University of Milano)
Alberto Parola (University of Insubria at Como)
Luca Dell'Anna, Giovanni Mazzarella, Flavio Toigo (University of Padova)

Summary

- Fermionic condensation in ultracold atoms
- Fermionic condensation in nuclear matter
- Fermionic condensation in neutron stars
- Conclusions
- Acknowledgments

Fermionic condensation in ultracold atoms (I)

The shifted Hamiltonian of the uniform two-spin-component Fermi superfluid made of ultracold atoms is given by

$$\hat{H}' = \int d^3\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^+_\sigma(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu \right) \hat{\psi}_\sigma(\mathbf{r}) + g\, \hat{\psi}^+_\uparrow(\mathbf{r})\, \hat{\psi}^+_\downarrow(\mathbf{r})\, \hat{\psi}_\downarrow(\mathbf{r})\, \hat{\psi}_\uparrow(\mathbf{r}) \, , \qquad (1)$$

where $\hat{\psi}_\sigma(\mathbf{r})$ is the field operator that annihilates a fermion of spin σ in the position $\mathbf{r}$, while $\hat{\psi}^+_\sigma(\mathbf{r})$ creates a fermion of spin σ in $\mathbf{r}$. Here $g < 0$ is the strength of the attractive fermion-fermion interaction.

Fermionic condensation in ultracold atoms (II)

The ground-state average of the number of fermions reads

$$N = \int d^3\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \langle \hat{\psi}^+_\sigma(\mathbf{r})\, \hat{\psi}_\sigma(\mathbf{r}) \rangle \ . \qquad (2)$$

This total number N is fixed by the chemical potential μ which appears in Eq. (1).
In a Fermi system the largest eigenvalue of the two-body density matrix gives the **number of Cooper pairs**, which is half of the number of condensed fermions N_0. Thus one finds[1]

$$N_0 = 2 \int d^3\mathbf{r}_1\, d^3\mathbf{r}_2\, |\langle \hat{\psi}_\downarrow(\mathbf{r}_1)\, \hat{\psi}_\uparrow(\mathbf{r}_2) \rangle|^2 \ . \qquad (3)$$

[1] A.J. Leggett, Quantum liquids. Bose condensation and Cooper pairing in condensed-matter systems (Oxford Univ. Press, Oxford, 2006)

Fermionic condensation in ultracold atoms (III)

Within the Bogoliubov approach the mean-field Hamiltonian derived from Eq. (1) can be diagonalized by using the Bogoliubov-Valatin representation of the field operator $\hat{\psi}_\sigma(\mathbf{r})$ in terms of the anticommuting quasi-particle Bogoliubov operators $\hat{b}_{\mathbf{k}\sigma}$ with amplitudes $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ and the quasi-particle energy $E_{\mathbf{k}}$. In this way one finds familiar expressions for these quantities:

$$E_{\mathbf{k}} = \left[(\epsilon_{\mathbf{k}} - \mu)^2 + \Delta^2 \right]^{1/2} \tag{4}$$

and

$$\begin{aligned} u_{\mathbf{k}}^2 &= \left(1 + (\epsilon_{\mathbf{k}} - \mu)/E_{\mathbf{k}}\right)/2 \qquad (5) \\ v_{\mathbf{k}}^2 &= \left(1 - (\epsilon_{\mathbf{k}} - \mu)/E_{\mathbf{k}}\right)/2 \, , \qquad (6) \end{aligned}$$

where $\epsilon_{\mathbf{k}} = \hbar^2 k^2/(2m)$ is the single-particle energy.

Fermionic condensation in ultracold atoms (IV)

The parameter Δ is the pairing gap, which satisfies the gap equation

$$-\frac{1}{g} = \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}}} \; , \tag{7}$$

where Ω is the volume of the uniform system. Notice that this equation is **ultraviolet divergent** and it must be regularized.
The equation for the total density $n = N/\Omega$ of fermions is obtained from Eq. (2) as

$$n = \frac{2}{\Omega} \sum_{\mathbf{k}} v_{\mathbf{k}}^2 \; . \tag{8}$$

Finally, from Eq. (3) one finds that the condensate density $n_0 = N_0/\Omega$ of paired fermions is given by[2]

$$n_0 = \frac{2}{\Omega} \sum_{\mathbf{k}} u_{\mathbf{k}}^2 v_{\mathbf{k}}^2 \; . \tag{9}$$

[2] L.S., N. Manini, A. Parola, PRA **72**, 023621 (2005).

Fermionic condensation in ultracold atoms (V)

In three dimensions, a suitable regularization[3] of the gap equation is obtained by introducing the s-wave scattering length a via the equation

$$-\frac{1}{g} = -\frac{m}{4\pi\hbar^2 a} + \frac{1}{\Omega}\sum_{\mathbf{k}} \frac{m}{\hbar^2 k^2}\,, \qquad (10)$$

and then subtracting this equation from the gap equation (7). In this way one obtains the three-dimensional **regularized gap equation**

$$-\frac{m}{4\pi\hbar^2 a} = \frac{1}{\Omega}\sum_{\mathbf{k}} \left(\frac{1}{2E_k} - \frac{m}{\hbar^2 k^2}\right), \qquad (11)$$

which can be used to study the full BCS-BEC crossover[4] by changing the amplitude and sign of the s-wave scattering length a.

[3] Marini, Pistolesi, and Strinati, Eur. Phys. J. B **1**, 151 (1998).

[4] D.M. Eagles, PR **186**, 456 (1969); A.J. Leggett, in *Modern Trends in the Theory of Condensed Matter*, p. 13, edited by A. Pekalski and J. Przystawa (Springer, Berlin, 1980).

Fermionic condensation in ultracold atoms (VI)

Taking into account the functional dependence of the amplitudes u_k and v_k on μ and Δ, one finds[5] the condensate density

$$n_0 = \frac{m^{3/2}}{8\pi\hbar^3}\,\Delta^{3/2}\sqrt{\frac{\mu}{\Delta}+\sqrt{1+\frac{\mu^2}{\Delta^2}}}\;. \tag{12}$$

By the same techniques, also the two BCS-BEC equations can be written in a more compact form as

$$-\frac{1}{a} = \frac{2(2m)^{1/2}}{\pi\hbar^3}\,\Delta^{1/2}\, I_1\!\left(\frac{\mu}{\Delta}\right)\;, \tag{13}$$

$$n = \frac{(2m)^{3/2}}{2\pi^2\hbar^3}\,\Delta^{3/2}\, I_2\!\left(\frac{\mu}{\Delta}\right)\;, \tag{14}$$

where $I_1(x)$ and $I_2(x)$ are two monotonic functions which can be expressed in terms of elliptic integrals[6].

[5] L.S., N. Manini, A. Parola, PRA **72**, 023621 (2005).
[6] Marini, Pistolesi, and Strinati, Eur. Phys. J. B **1**, 151 (1998).

Fermionic condensation in ultracold atoms (VII)

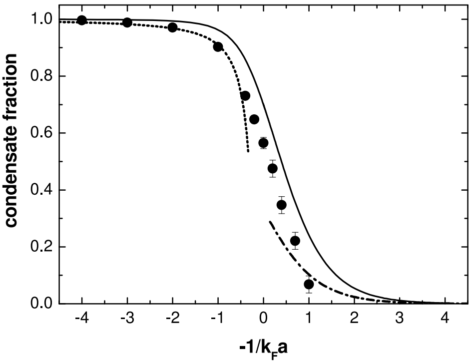


Figure: Condensate fraction of pairs as a function of the inverse interaction strength $y = 1/(k_F a)$: **our mean-field theory** (solid line); Fixed-Node Diffusion Monte Carlo results (symbols) [G. E. Astrakharchik et al., PRL **95**, 230405 (2005)]; Bogoliubov quantum depletion of a Bose gas with $a_m = 0.6a$ (dashed line); BCS theory (dot-dashed line).

Fermionic condensation in nuclear matter (I)

Let us now consider the **nuclear matter**, and in particular the neutron matter, which is a dense Fermi liquid made of two-component (spin up and down) neutrons. The shifted Hamiltonian of the uniform neutron matter can be written as

$$\begin{aligned}\hat{H}' &= \int d^3\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^+_\sigma(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu \right) \hat{\psi}_\sigma(\mathbf{r}) \\ &+ \int d^3\mathbf{r}\; d^3\mathbf{r}'\; \hat{\psi}^+_\uparrow(\mathbf{r})\; \hat{\psi}^+_\downarrow(\mathbf{r}')\; V(\mathbf{r}-\mathbf{r}')\; \hat{\psi}_\downarrow(\mathbf{r}')\; \hat{\psi}_\uparrow(\mathbf{r}) \; , \end{aligned} \tag{15}$$

where $\hat{\psi}_\sigma(\mathbf{r})$ is the field operator that annihilates a neutron of spin σ in the position $\mathbf{r}$, while $\hat{\psi}^+_\sigma(\mathbf{r})$ creates a neutron of spin σ in $\mathbf{r}$. Here $V(\mathbf{r}-\mathbf{r}')$ is the **neutron-neutron potential** characterized by s-wave scattering length $a = -18.5$ fm and effective range $r_e = 2.7$ fm.

Fermionic condensation in nuclear matter (II)

One can apply the familiar Bogoliubov approach to diagonalize the mean-field quadratic Hamiltonian derived from Eq. (16), but now the paring gap $\Delta_{\mathbf{k}}$ depends explicitly on the wave number $\mathbf{k}$ and satisfies the integral equation

$$\Delta_{\mathbf{q}} = \sum_{\mathbf{k}} V_{\mathbf{qk}} \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}} , \tag{16}$$

where

$$V_{\mathbf{qk}} = \langle \mathbf{q}, -\mathbf{q} | V | \mathbf{k}, -\mathbf{k} \rangle \tag{17}$$

is the wave-number representation of the neutron-neutron potential, and

$$E_{\mathbf{k}} = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu \right)^2 + |\Delta_{\mathbf{k}}|^2} . \tag{18}$$

Fermionic condensation in nuclear matter (III)

Under the simplifying assumptions

$$\mu \simeq \epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}\,, \qquad \Delta_{\mathbf{k}} \simeq \Delta\,, \tag{19}$$

in the continuum limit the gap equation of the neutron matter becomes

$$1 = \frac{1}{2}\int \frac{d^3\mathbf{k}\ d^3\mathbf{r}}{(2\pi)^3}\,\frac{V(\mathbf{r})\ e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{(\frac{\hbar^2k^2}{2m}-\epsilon_F)+\Delta^2}}\,. \tag{20}$$

Moreover, the number equation reads

$$n = \frac{1}{2}\frac{(2m)^{3/2}}{2\pi^2\hbar^3}\,\Delta^{3/2}\,I_2\left(\frac{\epsilon_F}{\Delta}\right)\,, \tag{21}$$

where $I_2(x)$ is the monotonic function

$$I_2(x) = \int_0^{+\infty} y^2\left(1-\frac{y^2-x}{\sqrt{(y^2-x)^2+1}}\right)dy\,. \tag{22}$$

Fermionic condensation in nuclear matter (IV)

In a similar way one gets the condensate density of neutron-neutron pairs

$$n_0 = \frac{m^{3/2}}{8\pi\hbar^3}\,\Delta^{3/2}\sqrt{\frac{\epsilon_F}{\Delta}+\sqrt{1+\frac{\epsilon_F^2}{\Delta^2}}}\;. \tag{23}$$

These equations show that knowing the scaled energy gap Δ/ϵ_F one can determine the condensate fraction

$$\frac{n_0}{n} = \frac{\pi}{2^{5/2}}\,\frac{\sqrt{\frac{\epsilon_F}{\Delta}+\sqrt{1+\frac{\epsilon_F^2}{\Delta^2}}}}{I_2(\frac{\epsilon_F}{\Delta})} \tag{24}$$

Notice that in the deep BCS regime where $\Delta/\epsilon_F \ll 1$ one finds

$$\frac{n_0}{n} = \frac{3\pi}{8}\,\frac{\Delta}{\epsilon_F}\;. \tag{25}$$

Fermionic condensation in nuclear matter (V)

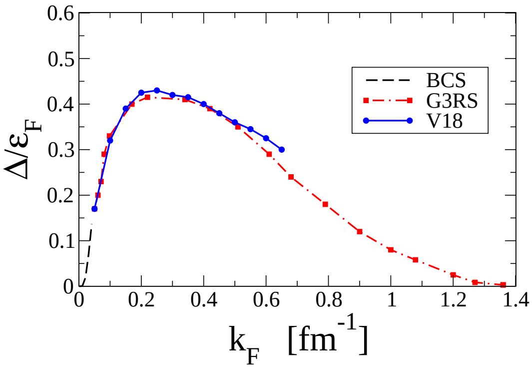


Figure: Scaled pairing gap Δ/ϵ_F vs Fermi wave number k_F. Dashed line: BCS limit; filled squares: results obtained with the G3RS nuclear potential [M. Matsuo, PRC **73**, 044309 (2006)]; filled circles: results obtained with the Argone V18 nuclear potential [A. Gezerlis and J. Carlson, PRC **81**, 025803 (2010)].

Fermionic condensation in nuclear matter (VI)

Fitting the Matsuo data[7] of Δ/ϵ_F vs k_F we obtain the formula[8]

$$\frac{\Delta}{\epsilon_F} = \frac{\beta_0 k_F^{\beta_1}}{\exp(k_F^{\beta_2}/\beta_3) - \beta_3} \tag{26}$$

with the following fitting parameters:
$\beta_0 = 2.851$, $\beta_1 = 1.942$, $\beta_2 = 1.672$, $\beta_3 = 0.276$, $\beta_4 = 0.975$.
By using this fitting formula and the simple equation

$$\frac{n_0}{n} = \frac{\pi}{2^{5/2}} \frac{\sqrt{\frac{\epsilon_F}{\Delta} + \sqrt{1 + \frac{\epsilon_F^2}{\Delta^2}}}}{I_2(\frac{\epsilon_F}{\Delta})} \tag{27}$$

we get the condensate fraction of neutron matter as a function of the neutron density n.

[7] M. Matsuo, PRC **73**, 044309 (2006).
[8] L.S., PRC **84**, 067301 (2011)

Fermionic condensation in nuclear matter (VII)

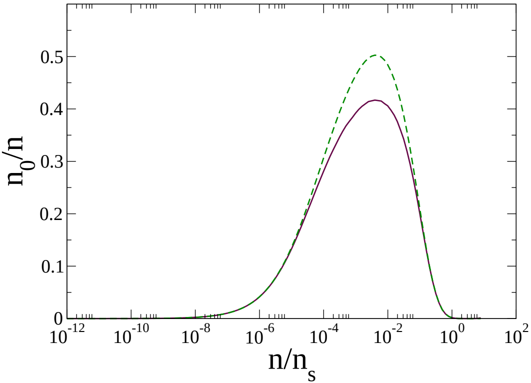
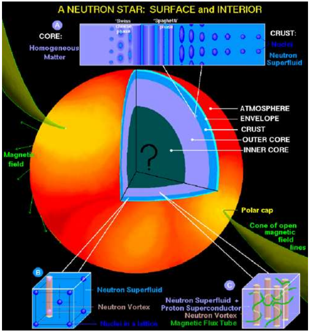


Figure: Condensate fraction n_0/n of neutron pairs in neutron matter as a function of the scaled density n/n_s, where $n_s = 0.16$ fm^{-3} is the nuclear saturation density. The solid line is obtained by using Eqs. (24) and (26). The dashed line is obtained by using Eqs. (25) and (26).

Fermionic condensation in neutron stars (I)

Neutron stars are astronomical compact objects that can result from the gravitational collapse of a massive star during a supernova event. Such stars are mainly composed of neutrons.



Neutron stars are very hot and are supported against further collapse by Fermi pressure. A typical neutron star has a mass M between 1.35 and about 2.0 solar masses with a corresponding radius R of about 12 km.

Fermionic condensation in neutron stars (II)

Notice that in the crust of neutron stars one estimates[9] $T \simeq 10^8$ K, while $T_c \simeq 10^{10}$ K. Thus the crust of neutron stars is superfluid.
In previous slides we have found a fitting formula for the condensate fraction n_0/n of neutron matter as a function of the Fermi wave number

$$k_F = (3\pi^2 n)^{1/3} \; . \tag{28}$$

Knowing the density profile $n(r)$ of a neutron star[10], i.e. the neutron density n as a function of the distance r from the center of a neutron star, we can determine[11] the condensate fraction n_0/n of the neutron star as a function of the distance r.

[9] S. Zane, R. Turolla, and D. Page, Isolated Neutron Stars: from the Surface to the Interior (Springer, Berlin, 2007).

[10] B. Datta, A.V. Thampan, and D. Bhattacharya, J. Astrophys. Astr. **16**, 375 (1995).

[11] LS, in preparation

Fermionic condensation in neutron stars (III)

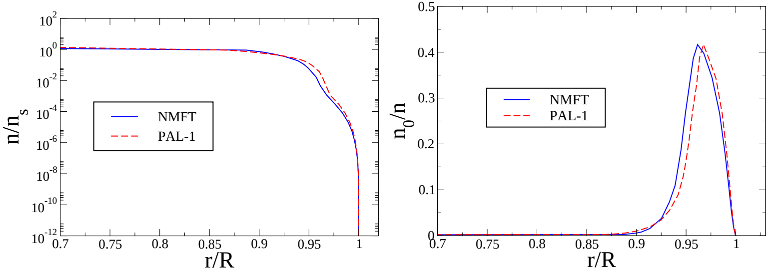


Figure: 1.4 solar mass neutron star. Left panel: Scaled density profile n/n_s vs scaled distance r/R. $n_s = 0.16$ fm^{-3} is the nuclear saturation density and R is the radius of the star. Right panel: condensate fraction n_0/n of neutron pairs vs scaled distance r/R. Solid line is a simple neutron matter model [J.D. Walecka, Ann. Phys. **83**, 491 (1974)]. Dashed line is a more realistic model [T.L. Ainsworth, and J.M. Lattimer, PRL **61**, 2518 (1988)].

Conclusions

We have seen that the condensate fraction of Cooper pairs can be calculated in various superfluid fermionic systems: dilute atomic gases, dense neutron matter and neutron stars.

Our results on these and similar topics are published in

L.S., N. Manini, and A. Parola, PRA **72**, 023621 (2005).
L.S, PRA **76**, 015601 (2007).
L.S., PRA **83**, 033630 (2011).
L. Dell'Anna, G. Mazzarella, and L.S., PRA **84**, 033633 (2011).
L.S., PRC **84**, 067301 (2011).
L. Dell'Anna, G. Mazzarella, and L.S., PRA **86**, 053632 (2012).
L.S., PRA **86**, 055602 (2012).
L.S. and F. Toigo, PRA **86**, 023619 (2012).

Acknowledgments

THANK YOU FOR YOUR ATTENTION!

We acknowledge research grants from:
– Università di Padova: Progetto di Ricerca di Ateneo 2012-2013.
– Fondazione CARIPARO: Progetto di Eccellenza 2012-2013.
– MIUR: Progetto PRIN 2012-2014.