# Regularization of quantum fluctuations in the BCS-BEC crossover

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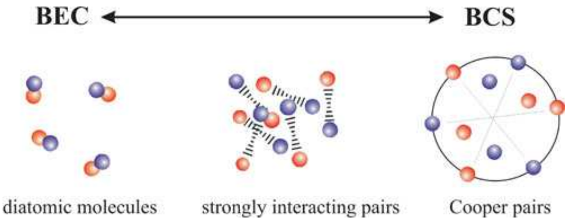
# Summary

- The BCS-BEC crossover
- Quantum fluctuations in $D$-dimensions
- Regularization of the 3D model
- Regularization of the 2D model
- Conclusions

# The BCS-BEC crossover (I)

In 2004 the 3D BCS-BEC crossover has been observed with **ultracold gases made of fermionic $^{40}$K and $^{6}$Li alkali-metal atoms.**[1]




This crossover is obtained by changing (with a Feshbach resonance) the s-wave scattering length $a_F$ of the inter-atomic potential:

– $a_F \to 0^-$ (BCS regime of weakly-interacting Cooper pairs)

– $a_F \to \pm\infty$ (unitarity limit of strongly-interacting Cooper pairs)

– $a_F \to 0^+$ (BEC regime of bosonic dimers)

[1] C.A. Regal et al., PRL **92**, 040403 (2004); M.W. Zwierlein et al., PRL **92**, 120403 (2004); M. Bartenstein, A. Altmeyer et al., PRL **92**, 120401 (2004); J. Kinast et al., PRL **92**, 150402 (2004).

# The BCS-BEC crossover (II)

The crossover from a BCS superfluid ($a_F < 0$) to a BEC of molecular pairs ($a_F > 0$) has been investigated experimentally around a Feshbach resonance, where the s-wave scattering length $a_F$ diverges, and it has been shown that the system is (meta)stable.
The detection of quantized vortices under rotation[2] has clarified that this dilute gas of ultracold atoms is superfluid.
Usually the BCS-BEC crossover is analyzed in terms of

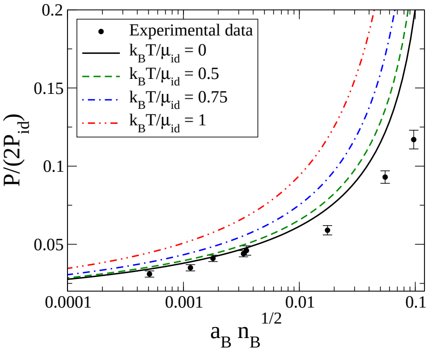
$$y = \frac{1}{k_F a_F} \tag{1}$$

the inverse scaled interaction strength, where $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wave number and $n$ the total density.
The system is dilute because $r_e k_F \ll 1$, with $r_e$ the effective range of the inter-atomic potential.

[2] M.W. Zwierlein *et al.*, Science **311**, 492 (2006); M.W. Zwierlein *et al.*, Nature **442**, 54 (2006)

# The BCS-BEC crossover (III)

In 2014 also the 2D BEC-BEC crossover has been achieved[3] with a **quasi-2D Fermi gas of $^6$Li atoms** with widely tunable s-wave interaction, measuring the pressure $P$ *vs* the gas parameter $a_B n^{1/2}$.




Filled circles with error bars are **experimental data** while solid lines are obtained with our beyond-mean-field finite-temperature theory[4].

[3] V. Makhalov, K. Martiyanov, and A. Turlapov, PRL **112**, 045301 (2014).
[4] LS and F. Toigo, "Zero-point energy of ultracold atoms", in preparation.

# Quantum fluctuations in $D$-dimensions (I)

We adopt the formalism of functional integration[5]. The partition function $\mathcal{Z}$ of the uniform system with fermionic fields $\psi_s(\mathbf{r},\tau)$ at temperature $T$, in a $D$-dimensional volume $L^D$, and with chemical potential $\mu$ reads

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \; \exp\left\{ -\frac{S}{\hbar} \right\} , \tag{2}$$

where ($\beta \equiv 1/(k_B T)$ with $k_B$ Boltzmann's constant)

$$S = \int_0^{\hbar\beta} d\tau \int_{L^D} d^D\mathbf{r} \; \mathcal{L} \tag{3}$$

is the Euclidean action functional with Lagrangian density

$$\mathcal{L} = \bar{\psi}_s \left[ \hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \psi_s + \mathbf{g}\, \bar{\psi}_\uparrow \, \bar{\psi}_\downarrow \, \psi_\downarrow \, \psi_\uparrow \tag{4}$$

where $\mathbf{g}$ is the attractive strength ($\mathbf{g} < 0$) of the s-wave coupling.

[5] N. Nagaosa, Quantum Field Theory in Condensed Matter Physics (Springer, 1999)

# Quantum fluctuations in $D$-dimensions (II)

Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathcal{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(\mathbf{r},\tau)$. In this way the effective Euclidean Lagrangian density reads

$$\mathcal{L}_e = \bar{\psi}_s \left[\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu\right] \psi_s + \bar{\Delta}\, \psi_\downarrow\, \psi_\uparrow + \Delta \bar{\psi}_\uparrow\, \bar{\psi}_\downarrow - \frac{|\Delta|^2}{\mathbf{g}} \; . \qquad (5)$$

We investigate the effect of fluctuations of the gap field $\Delta(\mathbf{r},t)$ around its mean-field value $\Delta_0$ which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r},\tau) = \Delta_0 + \eta(\mathbf{r},\tau) \; , \qquad (6)$$

where $\eta(\mathbf{r},\tau)$ is the complex field which describes pairing fluctuations.

# Quantum fluctuations in $D$-dimensions (III)

In particular, we are interested in the grand potential $\Omega$, given by

$$\Omega = -\frac{1}{\beta} \ln\left(\mathcal{Z}\right) \simeq -\frac{1}{\beta} \ln\left(\mathcal{Z}_{mf} \mathcal{Z}_g\right) = \Omega_{mf} + \Omega_B \ , \tag{7}$$

where $\mathcal{Z}_{mf}$ is the mean-field partition function and $\mathcal{Z}_g$ is the partition function of Gaussian bosonic pairing fluctuations.
To make a long story short, one finds that in the gas of paired fermions there are two kinds of elementary excitations: fermionic single-particle excitations with energy

$$E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2} \ , \tag{8}$$

where $\Delta_0$ is the pairing gap, and bosonic collective excitations which can be approximated by the low-momentum energy

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda \ \frac{\hbar^2 q^2}{2m} + 2 \ m \ c_s^2\right)} \ , \tag{9}$$

where $\lambda$ is the first correction to the familiar low-momentum phonon dispersion $E_{col}(q) \simeq c_s \hbar q$ and $c_s$ is the sound velocity.

# Quantum fluctuations in $D$-dimensions (IV)

At zero temperature the total grand potential reads

$$\Omega = \Omega_{mf} + \Omega_B \ , \tag{10}$$

where

$$\Omega_{mf} = -\frac{\Delta_0^2}{\mathbf{g}} L^2 - \sum_{\mathbf{k}} E_{sp}(k) \tag{11}$$

includes the zero-point energy of fermionic single-particle excitations, and

$$\Omega_B = \frac{1}{2} \sum_{\mathbf{q}} E_{col}(q) \tag{12}$$

is the zero-point energy of Gaussian bosonic collective excitations.
The **zero-point energy** due to fermionic and bosonic excitations

$$E_{zero} = -\sum_{\mathbf{k}} E_{sp}(k) + \frac{1}{2} \sum_{\mathbf{q}} E_{col}(q) \tag{13}$$

is ultraviolet divergent in any dimension $D$. These divergences are a consequence of the contact interaction we are using (neglecting finite size of atoms and finite range of interaction).

# Regularization of the 3D model (I)

The divergent **zero-point energy** $E_{zero}$ can be regularized by using different procedures (dimensional regularization, cut-off regularization, convergence factors regularization)[6] which give the same finite result.

For instance, in the deep BEC regime of the **3D BCS-BEC crossover**, where the fermionic scattering length $a_F$ becomes positive and the chemical potential $\mu$ becomes negative, performing

regularization of zero-point fluctuations

we have recently found[7] that the zero-temperature grand potential becomes

$$\Omega = -L^3 \frac{(1+\alpha)}{256\pi} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{\Delta_0^4}{|\mu|^{3/2}} , \tag{14}$$

with $\alpha = 2$ due to zero-point bosonic fluctuations.

[6] LS and F. Toigo, "Zero-point energy of ultracold atoms", in preparation

[7] LS and G. Bighin, PRA **91**, 033610 (2015).

# Regularization of the 3D model (II)

Taking into account Eq. (14), one easily obtains

$$\mu = -\frac{\hbar^2}{2m{a_F}^2} + \frac{\pi\hbar^2}{m}\frac{a_F}{(1+\alpha)}n \, , \tag{15}$$

where the second term is half of the chemical potential $\mu_B = 4\pi\hbar^2 a_B n_B/m_B$ of composite bosons of mass $m_B = 2m$, density $n_B = n/2$, and boson-boson scattering length

$$\boxed{a_B = \frac{2}{(1+\alpha)}a_F = \frac{2}{3}a_F \; .} \tag{16}$$

This analytical result[8] is in good agreement with numerical beyond-mean-field theoretical predictions.[9]

[8] LS and G. Bighin, Phys. Rev. A **91**, 033610 (2015).

[9] P. Pieri and G. Strinati, PRB **61**, 15370 (2000); D.S. Petrov, C. Salomon, and G.V. Shlyapnikov, PRL **93**, 090404 (2004); H.Hu, X.-J. Liu, and P. Drummond, EPL **74**, 574 (2006); R.B. Diener, R. Sensarma, and M. Randeria, PRA **77**, 023626 (2008).

# Regularization of the 2D model (I)

Similarly, in the deep BEC regime of the **2D BCS-BEC crossover**, where the chemical potential $\mu$ becomes negative, performing

regularization of zero-point fluctuations

we have recently found[10] that the zero-temperature grand potential is

$$\Omega = -\frac{mL^2}{64\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_b)^2 \ \ln\left(\frac{\epsilon_b}{2(\mu + \frac{1}{2}\epsilon_b)}\right) . \quad (17)$$

This is exactly Popov's equation of state of 2D Bose gas with chemical potential $\mu_B = 2(\mu + \epsilon_b/2)$ and mass $m_B = 2m$. In this way we have identified the two-dimensional scattering length $a_B$ of composite bosons as

$$\boxed{a_B = \frac{1}{2^{1/2}e^{1/4}} \ a_F \ .} \quad (18)$$

The value $a_B/a_F = 1/(2^{1/2}e^{1/4}) \simeq 0.551$ is in full agreement with $a_B/a_F = 0.55(4)$ obtained by Monte Carlo calculations[11].

[10] LS and F. Toigo, PRA **91**, 011604(R) (2015).

[11] G. Bertaina and S. Giorgini, PRL **106**, 110403 (2011).

# Conclusions

- The $D$-dimensional superfluid Fermi gas in the BCS-BEC crossover has a divergent zero-point energy.
- This divergent zero-point energy is due to both fermionic single-particle excitations and bosonic collective excitations.
- The regularization of zero-point energy gives remarkable analytical results for composite bosons in three dimensions[12] and in two dimensions[13].
- The final convergent equation of state is independent of the regularization procedure (dimensional regularization, cut-off regularization, convergence-factor regularization)[14], but it clearly depends on the dimensionality of the system and the two-dimensional case is highly nontrivial.

[12] LS and G. Bighin, PRA **91**, 033610 (2015).
[13] LS and F. Toigo, PRA **91**, 011604(R) (2015).
[14] LS and F. Toigo, "Zero-point energy of ultracold atoms", in preparation