Finite Size Effects in a Bosonic Josephson Junction

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Natal, Fabruary 4, 2021 [online]

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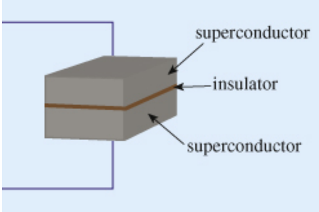
Summary

- Discovery of the Josephson effect and beyond
- Two site Bose-Hubbard model
- Standard mean-field dynamics
- Finite-size effects
- Numerical results
- Conclusions

Discovery of the Josephson effect and beyond (I)

In 1962 Brian Josephson predicted the mathematical relationship between the electric current and the electric voltage in a system made of two superconductors coupled with a thin insulating barrier.[1]



Josephson suggested that in this system it is possible the coherent macoscopic quantum tunneling of a large amount of superconducting Cooper pairs. A Cooper pair is a sort of bosonic-like particle made of two electrons with opposite spins and linear momenta.

The predictions of Josephson were quickly confirmed experimentally and he got the Nobel prize in Physics in 1973.

[1] B. D. Josephson, Phys. Lett. **1**, 251 (1962).

Discovery of the Josephson effect and beyond (II)

The Josephson junction can give rise to the direct-current (DC) Josephson effect, where a supercurrent flows indefinitely long across the barrier, but also to the alternate-current (AC) Josephson effect, where due to an energy difference the supercurrent oscillates periodically across the barrier[2].

The superconducting quantum interference devices (SQUIDs), which are very sensitive magnetometers based on superconducting Josephson junctions, are now widely used in science and engineering[3].

Josephson junctions are now also used to realize qubits of quantum computers[4].

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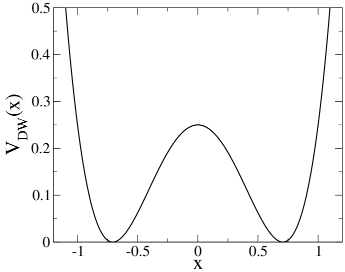
[2] A. Barone and G. Paterno, *Physics and Applications of the Josephson effect* (Wiley, 1982).

[3] E. L. Wolf, G.B. Arnold, M.A. Gurvitch, and John F. Zasadzinski, *Josephson Junctions: History, Devices, and Applications* (Pan Stanford Publishing, 2017).

[4] I. Buluta *et al.*, Rep. Prog. Phys. 74, 104401 (2011).

Discovery of the Josephson effect and beyond (III)

Bose-Einstein condensation (BEC) with ultracold and dilute alkali-metal atoms has increased the interest on the Josephson effect[5].
Bosons are confined by a double-well potential $V_{DW}(x)$ in the x axis and a very strong harmonic potential in the (y, z) plane.



We suppose that the barrier of the double-well potential $V_{DW}(x)$, with its maximum located at $x = 0$, is quite high such that there several doublets of quasi-degenerate single-particle energy levels.

[5] A. Smerzi *et al.*, Phys. Rev. Lett. **79**, 4950 (1997); M. Albiez *et al.*, Phys. Rev. Lett. **95**, 010402 (2005); G. Valtolina *et al.*, Science **350**, 1505 (2015).

Two-site Bose-Hubbard model (I)

The simplest quantum Hamiltonian of a system made of bosonic particles which are tunneling between two sites ($j = 1, 2$) is given by

$$\boxed{\hat{H} = -J\left(\hat{a}_1^+ \hat{a}_2 + \hat{a}_2^+ \hat{a}_1\right) + U \sum_{j=1,2} \hat{N}_j(\hat{N}_j - 1)} \, , \tag{1}$$

where $\hat{a}_j$ and $\hat{a}_j^+$ are the dimensionless ladder operators which, respectively, destroy and create a boson in the j site, $\hat{N}_j = \hat{a}_j^+ \hat{a}_j$ is the number operator of bosons in the j site. U is the on-site interaction strength of particles and $J > 0$ is the tunneling energy. Eq. (1) is the so-called two-site Bose-Hubbard Hamiltonian. We also introduce the total number operator

$$\hat{N} = \hat{N}_1 + \hat{N}_2 \, . \tag{2}$$

Two-site Bose-Hubbard model (II)

The time evolution of a generic quantum state $|\psi(t)\rangle$ of our system described by the Hamiltonian (1) is then given by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \ . \tag{3}$$

Quite remarkably, this time-evolution equation can be derived by extremizing the following action

$$S = \int dt \, \langle\psi(t)| \left(i\hbar \frac{\partial}{\partial t} - \hat{H} \right) |\psi(t)\rangle \ , \tag{4}$$

characterized by the Lagrangian

$$\boxed{L = i\hbar \langle\psi(t)| \frac{\partial}{\partial t}|\psi(t)\rangle - \langle\psi(t)|\hat{H}|\psi(t)\rangle} \ . \tag{5}$$

Standard mean-field dynamics (I)

The familiar mean-field dynamics of the bosonic Josephson junction can be obtained with a specific choice for the quantum state $|\psi(t)\rangle$, namely

$$|\psi(t)\rangle = |CS(t)\rangle \ , \tag{6}$$

where

$$|CS(t)\rangle = |\alpha_1(t)\rangle \otimes |\alpha_2(t)\rangle \tag{7}$$

is the tensor product of Glauber coherent states[6] $|\alpha_j(t)\rangle$ is such that

$$\hat{a}_j|\alpha_j(t)\rangle = \alpha_j(t)|\alpha_j(t)\rangle \ . \tag{8}$$

The complex eigenvalue $\alpha_j(t)$ can be written as

$$\alpha_j(t) = \sqrt{N_j(t)}\, e^{i\phi_j(t)} \ , \tag{9}$$

with $N_j(t) = \langle\alpha_j(t)|\hat{N}_j|\alpha_j(t)\rangle$ the average number of bosons in the site j at time t and $\phi_j(t)$ the corresponding phase angle at the same time t.

[6] R. Glauber, Phys. Rev. **131**, 2766 (1963).

Standard mean-field dynamics (II)

Adopting the coherent state (7) the Lagrangian (5) becomes

$$\boxed{L_{CS} = N\hbar\, z\dot{\phi} - \frac{UN^2}{2} z^2 + JN\sqrt{1-z^2}\,\cos(\phi)}\,, \tag{10}$$

where the dot means the derivative with respect to time t,

$$N = N_1(t) + N_2(t) \tag{11}$$

is the average total number of bosons (that is a constant of motion),

$$\phi(t) = \phi_2(t) - \phi_1(t) \tag{12}$$

is the relative phase, and

$$z(t) = \frac{N_1(t) - N_2(t)}{N} \tag{13}$$

is the population imbalance.

Standard mean-field dynamics (III)

In the Lagrangian $L_{CS}(\phi, z)$ of Eq. (10) the dynamical variables $\phi(t)$ and $z(t)$ are the generalized Lagrangian coordinates. The corresponding Euler-Lagrange equations are

$$\frac{\partial L_{CS}}{\partial \phi} - \frac{d}{dt}\frac{\partial L_{CS}}{\partial \dot{\phi}} = 0 \, , \tag{14}$$

$$\frac{\partial L_{CS}}{\partial z} - \frac{d}{dt}\frac{\partial L_{CS}}{\partial \dot{z}} = 0 \, , \tag{15}$$

which, explicitly, become

$$\dot{\phi} = J\frac{z}{\sqrt{1-z^2}}\cos(\phi) + UNz \, , \tag{16}$$

$$\dot{z} = -J\sqrt{1-z^2}\sin(\phi) \, . \tag{17}$$

These equations describe the mean-field dynamics of the macroscopic quantum tunneling in a Josephson junction[7].

[7] B. D. Josephson, Phys. Lett. **1**, 251 (1962); A. Smerzi *et al.*, Phys. Rev. Lett. **79**, 4950 (1997).

Standard mean-field dynamics (IV)

Assuming that both $\phi(t)$ and $z(t)$ are small, i.e. $|\phi(t)| \ll 1$ and $|z(t)| \ll 1$, one immediately gets the linearized Josephson-junction equations

$$\begin{aligned} \hbar\,\dot{\phi} &= (J + UN)z \,, \qquad (18) \\ \hbar\,\dot{z} &= -J\phi \,, \qquad (19) \end{aligned}$$

which can be rewritten as a single equation for the harmonic oscillation of $\phi(t)$ and the harmonic oscillation of $z(t)$, given by

$$\ddot{\phi} + \Omega^2\,\phi = 0 \,, \qquad (20)$$

$$\ddot{z} + \Omega^2\,z = 0 \,, \qquad (21)$$

both with frequency

$$\boxed{\Omega = \frac{1}{\hbar}\sqrt{J^2 + NUJ}} \,, \qquad (22)$$

that is the mean-field frequency of macroscopic quantum oscillation in terms of tunneling energy J, interaction strength U, and number N of particles.

Standard mean-field dynamics (V)

It is straightforward to find that the conserved energy of the mean-field system described by Eqs. (16) and (17) is given by

$$E_{CS} = \frac{UN^2}{2} z^2 - JN\sqrt{1-z^2}\cos(\phi) . \qquad (23)$$

If the condition

$$E_{CS}(z(0), \phi(0)) > E_{CS}(0, \pi) \qquad (24)$$

is satisfied then $\langle z \rangle \neq 0$ since $z(t)$ cannot become zero during an oscillation cycle: macroscopic quantum self trapping (MQST)[8]. The condition to get MQST reads

$$\boxed{\Lambda = \frac{NU}{J} > \Lambda_{MQST} = \frac{1+\sqrt{1-z^2(0)}\cos(\phi(0))}{z(0)^2/2}} . \qquad (25)$$

[8] A. Smerzi, S. Fantoni, S. Giovanazzi, and S.R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997).

Standard mean-field dynamics (VI)

Under the condition $U < 0$, due to the nonlinear interaction there are stationary ground-state solutions that break the z-symmetry

$$z_{\pm} = \pm\sqrt{1 - \frac{1}{\Lambda^2}} \tag{26}$$

$$\phi_n = 2\pi n \tag{27}$$

where $n \in \mathbb{Z}$. Indeed, the spontaneous symmetry breaking (SSB) of the balanced ground state ($z = 0$, $\phi = 0$) appears for

$$\boxed{\Lambda = \frac{NU}{J} < \Lambda_{SSB} = -1} \, . \tag{28}$$

Finite-size effects (I)

Different results are obtained by choosing another quantum state $|\psi(t)\rangle$ in Eqs. (4) and (5). In this section, our choice for the quantum state $|\psi(t)\rangle$ is

$$|\psi(t)\rangle = |ACS(t)\rangle \ , \tag{29}$$

where

$$|ACS(t)\rangle = \frac{\left(\sqrt{\frac{1+z(t)}{2}}\hat{a}_1^+ + \sqrt{\frac{1-z(t)}{2}}\, e^{-i\phi(t)}\hat{a}_2^+\right)^N}{\sqrt{N!}}|0\rangle \tag{30}$$

is the atomic coherent state (ACS), also called SU(2) coherent state or Bloch state or angular momentum coherent state, with $|0\rangle$ the vacuum state.[9]

Contrary to the Glauber coherent state $|CS(t)\rangle$ of Eq. (7), the atomic coherent state of Eq. (30) is an eigenstate of the total number operator (2), i.e.

$$\hat{N}|ACS(t)\rangle = N|ACS(t)\rangle \ . \tag{31}$$

[9] F. T. Arecchi *et al.*, Phys. Rev. A **6**, 2211 (1972).

Finite-size effects (II)

Adopting the atomic coherent state (30) the Lagrangian (5) becomes[10]

$$\boxed{L_{ACS} = N\hbar\, z\dot{\phi} - \frac{UN^2}{2}\left(1 - \frac{1}{N}\right) z^2 + JN\sqrt{1 - z^2}\,\cos(\phi)}\,. \qquad (32)$$

Comparing this expression with the Lagrangian of the Glauber coherent state, Eq. (10), one immediately observes that the two Lagrangians become equal under the condition $N \gg 1$.
In other words, the term $(1 - \frac{1}{N})$ takes into account few-body effects, which become negligible only for $N \gg 1$.

[10] P. Buonsante *et al*, Phys. Rev. A **72**, 043620 (2005); F. Trimborn *et al.*, Phys. Rev. A 77, 043631 (2008).

Finite-size effects (III)

It is easy to write down the corresponding Josephson equations

$$\dot{\phi} = J\frac{z}{\sqrt{1-z^2}}\cos(\phi) + UN\left(1-\frac{1}{N}\right)z \; , \qquad (33)$$

$$\dot{z} = -J\sqrt{1-z^2}\sin(\phi) \; , \qquad (34)$$

which are derived as the Euler-Lagrange equations of the Lagrangian (32).
Assuming that both $\phi(t)$ and $z(t)$ are small, i.e. $|\phi(t)| \ll 1$ and $|z(t)| \ll 1$ one finds the linearized Josephson-junction equations

$$\hbar\,\dot{\phi} = \left(J + UN\left(1-\frac{1}{N}\right)\right)z \; , \qquad (35)$$

$$\hbar\,\dot{z} = -J\phi \; . \qquad (36)$$

Finite-size effects (IV)

The linearized Josephson-junction equations can be rewritten as a single equation for the harmonic oscillation of $\phi(t)$ and the harmonic oscillation of $z(t)$, given by

$$\ddot{\phi} + \Omega_A^2 \, \phi = 0 \, , \quad (37)$$

$$\ddot{z} + \Omega_A^2 \, z = 0 \, , \quad (38)$$

both with frequency

$$\boxed{\Omega_A = \frac{1}{\hbar}\sqrt{J^2 + NUJ\left(1 - \frac{1}{N}\right)}} \, , \quad (39)$$

that is the atomic-coherent-state frequency of macroscopic quantum oscillation in terms of tunneling energy J, interaction strength U, and number N of particles.[11]

[11] S. Wimberger, A. Brollo, G. Manganelli, and LS, arXiv:2101.01009.

Finite-size effects (V)

In the same fashion as in the previous sections, the conserved energy associated to Eqs. (33) and (34) reads

$$E_{ACS} = \frac{UN^2}{2}\left(1 - \frac{1}{N}\right) z^2 - JN\sqrt{1 - z^2}\,\cos(\phi) \tag{40}$$

and using the self-trapping condition we get the inequality

$$\boxed{\Lambda = \frac{NU}{J} > \Lambda_{MQST,A} = \frac{1 + \sqrt{1 - z^2(0)}\cos(\phi(0))}{z(0)^2/2}\frac{1}{\left(1 - \frac{1}{N}\right)}}\,, \tag{41}$$

where $\Lambda_{MQST,A}$ is the atomic-coherent-state MQST critical parameter in terms of tunneling energy J, interaction strength U, and number N of particles.[12]

[12] S. Wimberger, A. Brollo, G. Manganelli, and LS, arXiv:2101.01009.

Finite-size effects (VI)

In addition to the usual symmetric stationary solutions we obtain from the system of Eq. (33) and (34) a correction to the symmetry breaking ones. In particular, we find that for

$$\boxed{\Lambda = \frac{NU}{J} < \Lambda_{SSB,A} = -\frac{1}{1 - \frac{1}{N}}} \qquad (42)$$

the ground state is not balanced.[13]
Clearly, for $N \gg 1$ from Eq. (42) one gets Eq. (28), while for $N = 1$ one finds $\Lambda_{SSB,A} = -\infty$: with only one boson the spontaneous symmetry breaking cannot be obtained.

[13] S. Wimberger, A. Brollo, G. Manganelli, and LS, arXiv:2101.01009.

Numerical results (I)

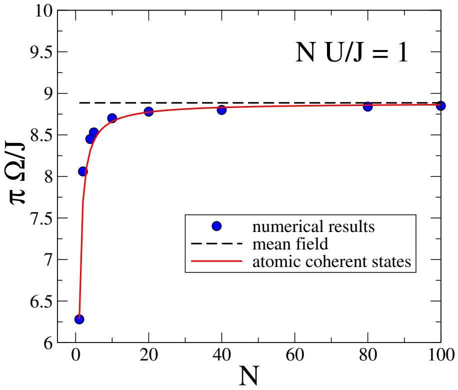
To test our analytical results we compare them with exact numerical simulations. The initial many-body state $|\Psi(0)\rangle$ for the time-dependent numerical simulations is the coherent state $|ACS(0)\rangle$ from Eq. (30), with a given choice of $z(0)$ and $\phi(0)$. The time evolved many-body state is then formally obtained as

$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\Psi(0)\rangle \ , \tag{43}$$

with $\hat{H}$ given by Eq. (1).
Knowing $|\Psi(t)\rangle$ the exact population imbalance at time t is given by

$$z_{ex}(t) = \langle\Psi(t)|\frac{\hat{N}_1 - \hat{N}_2}{N}|\Psi(t)\rangle \ . \tag{44}$$

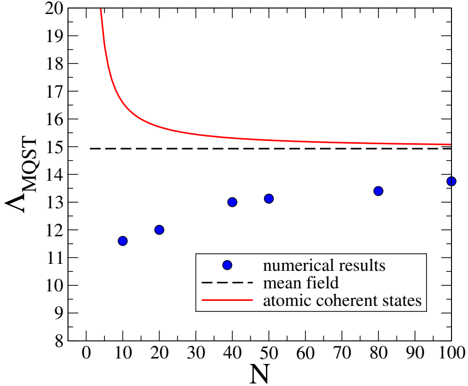
Numerical results (II)



Josephson frequency Ω as a function of the number N of bosons, with $UN/J = 1$ and $\hbar = 1$. Filled circles: exact numerical results. Dashed line: mean-field result, Eq. (22), based on Glauber coherent states. Solid curve: results of Eq. (39), based on atomic coherent states (ACS). Initial conditions: $z(0) = 0.1$ and $\phi(0) = 0$.[14]

[14] S. Wimberger, A. Brollo, G. Manganelli, and LS, arXiv:2101.01009.

Numerical results (III)



Critical interaction strength Λ_{MQST} for the macroscopic quantum self trapping (MQSF) as a function of the number N of bosons. Filled circles: exact numerical results. Dashed line: mean-field result, Eq. (25), based on Glauber coherent states. Solid curve: results of Eq. (41), based on atomic coherent states (ACS). Initial conditions: $z(0) = 0.5$ and $\phi(0) = 0$.[15]

[15] S. Wimberger, A. Brollo, G. Manganelli, and LS, arXiv:2101.01009.

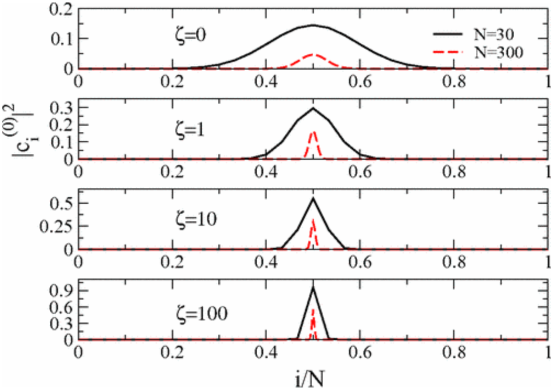
Numerical results (IV)

The exact number-conserving ground state of our system can be written as

$$|GS\rangle = \sum_{j=0}^{N} c_i^{(0)} \, |i\rangle_1 \otimes |N-i\rangle_2 \; , \tag{45}$$

where $|c_i^{(0)}|^2$ is the probability of finding the ground state with i bosons in the site 1 and $N-j$ bosons in the site 2. Here $|i\rangle_1$ is the Fock state with i bosons in the site 1 and $|N-i\rangle$ is the Fock state with $N-i$ bosons in the site 2. The amplitude probabilities c_i are determined numerically by diagonalizing the $(N+1)\times(N+1)$ Hamiltonian matrix obtained from (1).
Clearly these amplitude probabilities $c_i^{(0)}$ strongly depend on the values of the hopping parameter J, on-site interaction strength U, and total number N of bosons.

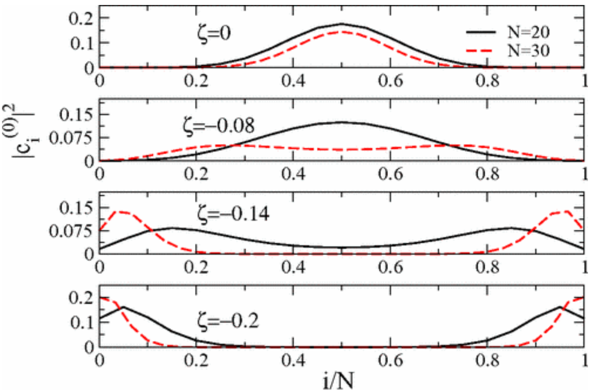
Numerical results (V)



Square modulus of the coefficients $c_i^{(0)}$ of the repulsive ($U > 0$) ground state. $|c_i^{(0)}|^2$ gives the probability of finding i bosons on the site 1 and $N - i$ bosons on the site 2. Here $\zeta = U/J$ and N is the total number of bosons.[16]

[16] G. Mazzarella, LS, A. Parola, and F. Toigo, Phys. Rev. A **83**, 053607 (2011).

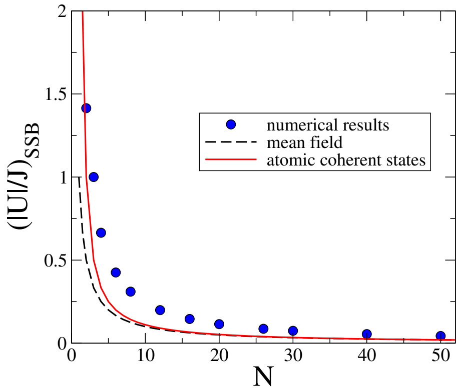
Numerical results (VI)



Square modulus of the coefficients $c_i^{(0)}$ of the attractive ($U < 0$) ground state. $|c_i^{(0)}|^2$ gives the probability of finding i bosons on the site 1 and $N - i$ bosons on the site 2. Here $\zeta = U/J$ and N is the total number of bosons.[17]

[17] G. Mazzarella, LS, A. Parola, and F. Toigo, Phys. Rev. A **83**, 053607 (2011).

Numerical results (VII)



Dimensionless interaction strength $(|U|/J)_{SSB}$ for the onset of spontaneous symmetry breaking (SSB) as a function of the number N of bosons. Filled circles: exact numerical results obtained from the onset of a bimodal structure in the distribution $\mathcal{P}(|c_i^{(0)}|^2)$. Dashed line: mean-field result based on Glauber coherent states. Solid curve: result based on atomic coherent states (ACS).[18]

[18] S. Wimberger, A. Brollo, G. Manganelli, and LS, arXiv:2101.01009.

Conclusions (I)

- We have obtained an analytical formula with $1/N$ corrections to the standard mean-field treatment for the frequency of Josephson oscillations. We have shown that this formula is in very good agreement with numerical calculations and it reduces to the familiar mean-field one in the large N limit.
- We have also investigated the spontaneous symmetry breaking of the ground state. Also in this case the agreement between the analytical predictions of the atomic coherent states and numerical results is good.
- We have studied the critical interaction strength for the macroscopic quantum self trapping. Here we have found that the $1/N$ corrections to the standard mean-field theory predicted by the atomic coherent states do not work compared with numerical simulations.

Conclusions (II)

- The time-dependent variational ansatz with atomic coherent states is quite reliable in the description of the short-time dynamics of the bosonic Josephson junction both in the Rabi regime, where $0 \leq U/J \ll 1/N$, and in the Josephson regime, where $1/N \ll U/J \ll N$.[19] Instead, in the Fock regime, where $U/J \gg N$, a full many-body quantum treatment is needed.
- We stress that experiments with cold atoms in lattices and double wells[20] show that atom numbers well below $N = 100$ can be reached and successfully detected with an uncertainty of the order one atom.

[19] A. J. Leggett, Rev. Mod. Phys. **73**, 307 (2001).

[20] C. Gross *et al.*, Nature **480**, 219 (2011); D. B. Hume *et al.*, Phys. Rev. Lett. **111**, 253001 (2013).

Acknowledgements

Thank you for your attention!