Collisionless Dynamics in Two-Dimensional Bosonic Gases

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Talk based on the paper
Summary

- 2D superfluids and BKT
- Recent experiment at ENS Paris
- 2D Boltzmann-Landau-Vlasov equation
- Collisionless sound and comparison with experiment
- Conclusions
BEC phase transition: For an ideal gas of non-interacting identical bosons there is Bose-Einstein condensation (BEC) only below a critical temperature $T_{BEC}$. In particular one finds

$$k_B T_{BEC} = \begin{cases} \frac{1}{2\pi \zeta(3/2)^{2/3}} \frac{\hbar^2}{m} n^{2/3} & \text{for } D = 3 \\ 0 & \text{for } D = 2 \\ \text{no solution} & \text{for } D = 1 \end{cases}$$

where $D$ is the spatial dimension of the system, $n$ is the number density, and $\zeta(x)$ is the Riemann zeta function.

This result due to Einstein (1925), which says that there is no BEC at finite temperature for $D \leq 2$ in the case of non-interacting bosons, was extended to interacting systems by Mermin and Wagner in 1966.
For the BEC phase transition, the Mermin-Wagner theorem\textsuperscript{1} states that there is no Bose-Einstein condensation at finite temperature in homogeneous systems with sufficiently short-range interactions in dimensions $D \leq 2$.

Despite the absence of BEC, Kosterlitz and Thouless\textsuperscript{2} (but also Berezinskii\textsuperscript{3}) suggested that a 2D fluid can be superfluid below a critical temperature, the so-called Berezinskii-Kosterlitz-Thouless (BKT) critical temperature $T_c$.

The Kosterlitz-Thouless (KT) transition has been observed experimentally in various physical systems: thin films of superfluid $^4$He (1978), thin films of superconductors (1981), quasi-2D bosonic gas of $^{87}$Rb atoms (2006). The KT transition has been observed also in quasi-2D superfluid fermionic gases made of $^6$Li atoms in the BCS-BEC crossover (2015).

\textsuperscript{3}V. L. Berezinskii, Sov. Phys. JETP \textbf{34}, 610 (1972).
An important prediction of the Kosterlitz-Thouless transition is that, contrary to the 3D case, the superfluid density $n_s(T)$ in 2D jumps to zero at the BKT critical temperature $T_c$. 
According to the two-fluid hydrodynamics of Landau-Khalatnikov the sound velocity has two branches: first sound velocity $c_1$ and second sound velocity $c_2$.

The slower one is the second sound velocity, which also jumps to zero at the BKT critical temperature $T_c$ for 2D bosonic gases.\textsuperscript{5}


Recently the speed of sound in a uniform quasi-2D Bose gas of $^{87}\text{Rb}$ atoms has been measured at Ecole Normale Superieure (ENS) of Paris$^6$.

These experimental results are in agreement with the predictions of the two-fluid hydrodynamics of Landau-Khalatnikov only well below the BKT critical temperature $T_c$.

The strong discrepancy near and above $T_c$ between the experimental data\textsuperscript{7} and the two-fluid hydrodynamics of Landau-Khalatnikov has been explained\textsuperscript{8} by suggesting that the experimental conditions are such that in this case collisions are not efficient enough to ensure the local thermodynamic equilibrium required by hydrodynamics and therefore the dynamics is collisionless.

Moreover, above the BKT critical temperature $T_c$ the experimental results show the presence of a strong damping of the sound mode.

In the next slides we reproduce all the experimental results found at ENS Paris by using the collisionless Boltzmann-Landau-Vlasov equation.

Due to the presence of a strong harmonic confinement along the $z$ axis, for the collisionless bosonic system one finds\(^9\) that the planar distribution $f(r, p)$ of atoms in the 4D single-particle phase space $(r, p) = (x, y, p_x, p_y)$ satisfies the 2D Boltzmann-Landau-Vlasov equation

$$\left[ \frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla r - \nabla r \left( U_{\text{mf}} \right) \cdot \nabla p \right] f(r, p, t) = 0 \tag{1}$$

where

$$U_{\text{mf}}(r, t) = g_{2D} \int \frac{d^2 p}{(2\pi\hbar)^2} f(r, p, t) \tag{2}$$

is the Hartree-Fock mean-field term and

$$g_{2D} = \frac{\sqrt{8\pi\hbar^2}}{m} \left( \frac{a_s}{a_z} \right) \tag{3}$$

with $a_s$ the 3D scattering length and $a_z$ the characteristic length of the harmonic confinement along the $z$ axis.

We set
\[ f(r, p, t) = f_0(p) + \delta f(r, p, t) \quad (4) \]
where \( f_0(p) \) is a stationary and isotropic distribution and \( \delta f(r, p, t) \) a very small perturbation around it.
It follows that the linearized equation for \( \delta f(r, p, t) \) reads
\[
\left[ \frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_r \right] \delta f(r, p, t) = g_{2D} \int \frac{d^2p'}{(2\pi \hbar)^2} \nabla_r \delta f(r, p', t) \cdot \nabla_p f_0(p) . \quad (5)
\]
Performing the Fourier transform of this equation according to
\[
\hat{\delta f}(k, p, \omega) = \int dt \int d^2r \delta f(r, p, t) \exp(i(k \cdot r - \omega t)) \text{ with } k \text{ the 2D wavevector and } \omega \text{ the angular frequency, one finds an implicit formula for the dispersion relation, given by}
\[
1 - g_{2D} \int \frac{d^2p}{(2\pi \hbar)^2} \frac{k \cdot \nabla_p f_0(p)}{p \cdot k/m - \omega} = 0 . \quad (6)
\]
In Eq. (6) there is a singularity on the integration path for \( \omega = \mathbf{p} \cdot \mathbf{k}/m \). In order to attach a meaning to the integral, we must interpret \( \omega \) as a complex quantity, i.e.

\[
\omega = \omega_R + i \omega_I, \tag{7}
\]

where \( \omega_I > 0 \) in order to avoid an exponential growth of the perturbation. Eq. (6) can be further simplified by assuming, without loss of generality, that \( \mathbf{k} = (k, 0) \). In this way one finds

\[
1 - g_{2D} \int \frac{dp_x}{(2\pi\hbar)} \frac{\partial \tilde{f}_0(p_x)}{\partial p_x} \frac{1}{p_x/m - c} = 0 \tag{8}
\]

where

\[
c = \frac{\omega}{k} \tag{9}
\]

is the **collisionless sound** and \( \tilde{f}_0(p_x) = \int f_0(p_x, p_y) dp_y/(2\pi\hbar) \).
From Eq. (8) one can extract the speed of sound $c$ in our collisionless regime. This velocity is, in general, a complex number such that

$$c = \frac{\omega}{k} = c_R + i c_I \quad (10)$$

with $c_R = \omega_R / k$ and $c_I = \omega_I / k$.

In the limit of weakly damped wave, i.e. $c_I \ll c_R$, one finds two coupled equations for $c_R$ and $c_I$:

$$1 - g_{2D} \mathcal{P} \int \frac{dp_x}{(2\pi \hbar)} \left[ \frac{\partial \tilde{f}_0(p_x)}{\partial p_x} \right]/\frac{p_x/m - c_R}{p_x/m - c_R} - \pi c_I \frac{\partial \phi(c)}{\partial c} \bigg|_{c_R} = 0 \quad (11)$$

$$c_I = \frac{\pi \frac{\partial \tilde{f}_0(p_x)}{\partial p_x} \bigg|_{p_x = mc_R}}{\frac{\partial}{\partial c_R} \left\{ \mathcal{P} \int \frac{dp_x}{(2\pi \hbar)} \left[ \frac{\partial \tilde{f}_0(p_x)}{\partial p_x} \right]/\frac{p_x/m - c_R}{p_x/m - c_R} \right\} \bigg), \quad (12)$$

where we denote $\phi(c) = \frac{mg_{2D}}{(2\pi \hbar)} \frac{\partial \tilde{f}_0}{\partial p_x} \bigg|_{p_x = mc}$ and $\mathcal{P}$ means principal value.
Collisionless sound and comparison with experiment (II)

We choose the Bose-Einstein distribution function

\[ f_0(p) = \frac{1}{L^2} \frac{1}{e^{(\frac{p^2}{2m} + g_2Dn - \mu)/(k_B T)} - 1} \approx \frac{1}{L^2} \left( \frac{k_B T}{\frac{p^2}{2m}} - \tilde{\mu} \right) \]  \hspace{1cm} (13)

as the equilibrium distribution of 2D weakly-interacting bosonic atoms

with \( n = N/L^2 \), where \( k_B \) is the Boltzmann constant and \( T \) is the absolute temperature. The Hartree interaction term \( g_2Dn \) can be formally removed by introducing a shifted chemical potential \( \tilde{\mu} = \mu - g_2Dn \).

From the normalization condition

\[ N = \int \frac{d^2r d^2p}{(2\pi \hbar)^2} f_0(p) , \]  \hspace{1cm} (14)

one finds the equation of state

\[ \tilde{\mu} = k_B T \ln \left( 1 - e^{-T_B/T} \right) \]  \hspace{1cm} (15)

where \( k_B T_B = 2\pi \hbar^2 n/m \) is the temperature of Bose degeneracy and clearly \( \tilde{\mu} < 0 \). The approximate expression for \( f_0(p) \) is valid under the condition \( T \ll T_B \) (degenerate regime).
In this way, the coupled equations for the real and imaginary part of the zero-sound velocity read

\[
1 + \tilde{g}_2 D k_B T \left[ \frac{2}{m c_R^2 - 2 \tilde{\mu}} + \frac{\sqrt{m c_R^2}}{(m c_R^2 - 2 \tilde{\mu})^{3/2}} \ln \left( \frac{\sqrt{m c_R^2 - 2 \tilde{\mu}} - \sqrt{m c_R^2}}{\sqrt{m c_R^2 - 2 \tilde{\mu}} + \sqrt{m c_R^2}} \right) \right]
+ \tilde{g}_2 D k_B T c I \frac{m c_R^2 + \tilde{\mu}}{\sqrt{m}(m c_R^2 - 2 \tilde{\mu})^{5/2}} = 0 ,
\]

\[(16)\]

where

\[
c_I = -\frac{\sqrt{m}}{6 c_R} \frac{c_R}{(m c_R^2 - 2 \tilde{\mu})^{3/2}} \left[ \frac{6 c_R}{(m c_R^2 - 2 \tilde{\mu})^2} + \frac{2(m c_R^2 + \tilde{\mu})}{\sqrt{m}} \sqrt{m c_R^2 - 2 \tilde{\mu} + \sqrt{m c_R^2}} \log \left( \frac{\sqrt{m c_R^2 - 2 \tilde{\mu}} - \sqrt{m c_R^2}}{\sqrt{m c_R^2 - 2 \tilde{\mu}} + \sqrt{m c_R^2}} \right) \right] ,
\]

\[(17)\]

where

\[
\tilde{g}_2 D = \frac{m}{\hbar^2} g_2 D = \sqrt{8 \pi} \frac{a_s}{a_z} .
\]

\[(18)\]
Sound velocity $c_R$ in units of $c_B = \sqrt{g_{2D} n / m}$ vs $T / T_c$ for $\tilde{g}_{2D} = 0.16$. The solid black line represents our theoretical prediction$^{10}$ while the blue dots are the experimental data obtained at ENS Paris.$^{11}$ The red dashed line is obtained by our equations with $c_I = 0$. Here $T_c = 0.13 T_B$.

Collisionless sound and comparison with experiment (V)

Imaginary part $c_I$ of the sound velocity in units of $c_B = \sqrt{g_{2D} n/m}$ as a function of the scaled temperature $T / T_c$ for $\tilde{g}_{2D} \simeq 0.16$. 

Inset: Ratio $c_I / c_R$ between the imaginary and the real part of the sound velocity $c$ as a function of the temperature.
Conclusions

- We have analyzed the sound propagation in collisionless bosonic gases assuming a 2D configuration.
- By solving the linearized 2D Boltzmann-Landau-Vlasov equation in the degenerate regime, we have:
  - derived two coupled algebraic equations for the real and imaginary part of the sound velocity;
  - compared our theoretical results with the recent experiment at ENS Paris;
  - found a very good agreement between our theory and the experiment.
- Our theoretical analysis strongly suggests that the density perturbation used in the experiment has excited the "bosonic zero sound", i.e. the sound of a collisionless bosonic fluid.
- The two-fluid hydrodynamics of Landau-Khalatnikov describes correctly only the collisional regime.
Thank you for attention!

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