Dynamical Josephson effect with superfluid Fermi atoms across a Feshbach resonance

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Partial support: CARIPARO Foundation and GNFM-INdAM

Talk at LPHYS'08, July 2, 2008

Summary

- Hydrodynamics of Fermi superfluids at zero-temperature
- Nonlinear Schrödinger equation for the BCS-BEC crossover
- Direct current Josephson effect
- Alternate current Josephson effect
- Conclusions

Hydrodynamics of Fermi superfluids at zero-temperature

At zero temperature, hydrodynamics equations of a two-component fermionic superfluid made of atoms of mass m are given by

$$\frac{\partial}{\partial t} n + \nabla \cdot (n\mathbf{v}) = 0 \tag{1}$$

$$m\frac{\partial}{\partial t}\mathbf{v} + \nabla \left[\frac{1}{2} m v^2 + U(\mathbf{r}) + \mu(n, a_F) \right] = 0 \tag{2}$$

where $n(\mathbf{r}, t)$ is the local density and $\mathbf{v}(\mathbf{r}, t)$ is the local superfluid velocity. Here $n(\mathbf{r}, t) = n_{\uparrow}(\mathbf{r}, t) + n_{\downarrow}(\mathbf{r}, t)$ with $n_{\uparrow}(\mathbf{r}, t) = n_{\downarrow}(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_{\uparrow}(\mathbf{r}, t) = \mathbf{v}_{\downarrow}(\mathbf{r}, t)$.

$U(\mathbf{r})$ is the external potential and $\mu(n, a_F)$ is the bulk chemical potential, i.e. the zero-temperature equation of state of the uniform system, which depends on the Fermi-Fermi scattering length a_F.

The density $n(\mathbf{r}, t)$ is such that

$$N = \int n(\mathbf{r}, t) \; d^3\mathbf{r} \tag{3}$$

is the total number of atoms in the fluid. In fact, due to the absence of the normal component, the superfluid density coincides with the total density and the superfluid current with the total current.

Hydrodynamics equations of superfluids are nothing else than the Euler equations of an inviscid (i.e. not-viscous) and irrotational fluid. The condition of irrotationality

$$\nabla \wedge \mathbf{v} = 0 \tag{4}$$

means that the velocity $\mathbf{v}$ can be written as the gradient of a scalar field. The connection between superfluid hydrodynamics and quantum mechanics is made by the formula

$$\mathbf{v} = \frac{\hbar}{2m} \nabla \theta \tag{5}$$

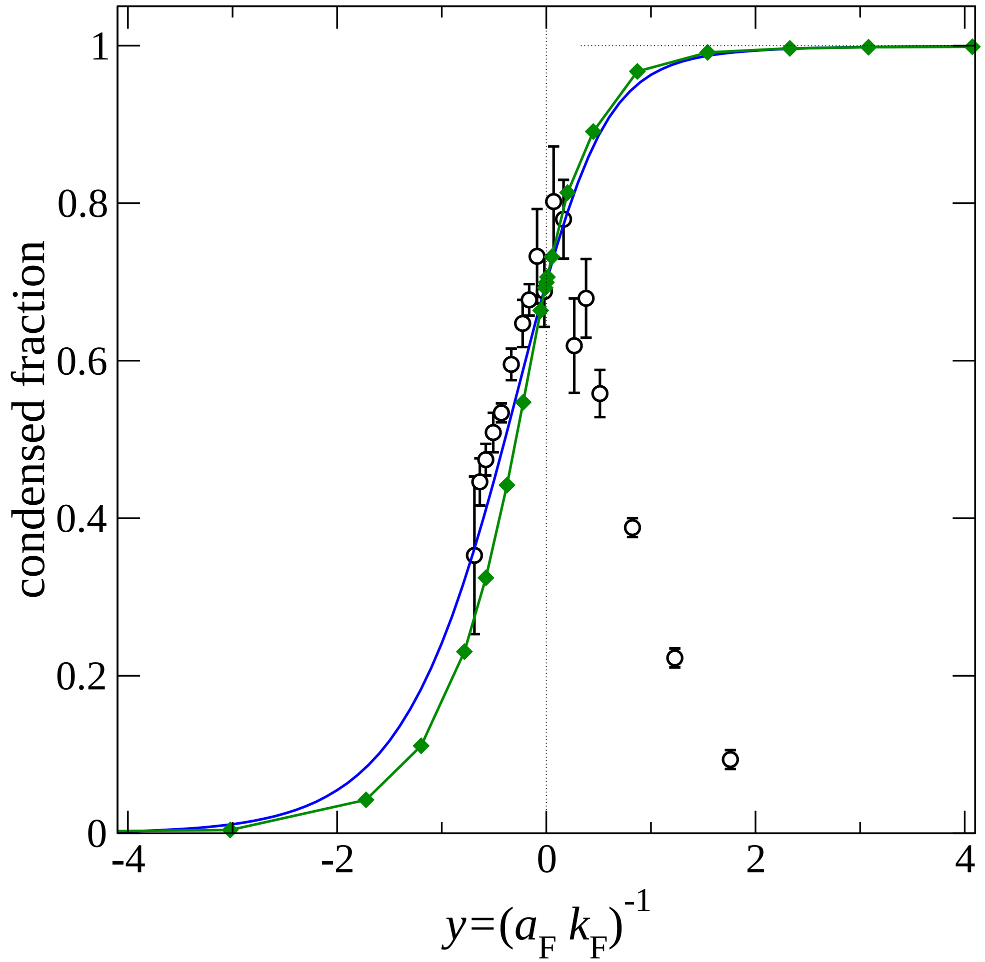
where $\theta(\mathbf{r}, t)$ is the phase of the condensate wave-function

$$\Xi(\mathbf{r}, t) = |\Xi(\mathbf{r}, t)| \; e^{i\theta(\mathbf{r},t)} = \langle \hat{\psi}_{\uparrow}(\mathbf{r}, t) \hat{\psi}_{\downarrow}(\mathbf{r}, t) \rangle \tag{6}$$

with $\hat{\psi}_{\sigma}(\mathbf{r}, t)$ the fermionic field operator with spin component $\sigma = \uparrow, \downarrow$. Notice $2m$ (Cooper pairs) instead of m in Eq. (5).

The condensate fraction of the Fermi superfluid is

$$\frac{N_0}{N} = \frac{2}{N} \int |\Xi(\mathbf{r}, t)|^2 \; d^3\mathbf{r} \tag{7}$$



Condensate fraction N_0/N of Fermi pairs in the BCS-BEC crossover as a function of the interse interaction parameter $y = (k_F a_F)^{-1}$ (solid line and joined diamonds). Open circles with error bars: the same quantity from the MIT experiment [M.W. Zwierlein *et al.*, PRL **94**, 180401 (2005)].

From L.S., N. Manini, A. Parola, PRA **72**, 023621 (2005).

In the BCS-BEC crossover the bulk chemical potential can be written as

$$\mu(n, a_F) = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}\left(f(y) - \frac{y}{5}f'(y)\right) \tag{8}$$

where $f(y)$ is a dimensionless universal function of the inverse interaction parameter

$$y = \frac{1}{k_F a_F} = \frac{1}{(3\pi^2 n)^{1/3} a_F} \tag{9}$$

with $k_F = (3\pi^2 n)^{1/3}$ the Fermi wavenumber and a_F the Fermi-Fermi scattering length.

We parametrize $f(y)$ as follows

$$f(y) = \alpha_1 - \alpha_2 \arctan\left(\alpha_3\, y\, \frac{\beta_1 + |y|}{\beta_2 + |y|}\right) \tag{10}$$

where the values of the parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ are reported in N. Manini and L.S., PRA **71**, 033625 (2005). This reliable fitting function is based on asymptotics and fixed-node Monte-Carlo data [G.E. Astrakharchik *et al.*, PRL **93**, 200404 (2004)].

Hydrodynamics equations of superfluids describe efficiently:
i) static properties;
ii) low-energy collective dynamics with $\lambda \gg \xi$,
where λ is the wavelength of the mode, ξ is healing length of the superfluid (e.g. they give the correct Bogoliubov-Anderson-Goldstone mode).
Combescot, Kagan and Stringari [PRA **74**, 042717 (2006)] suggest

$$\xi = \frac{\hbar}{m v_{cr}} \tag{11}$$

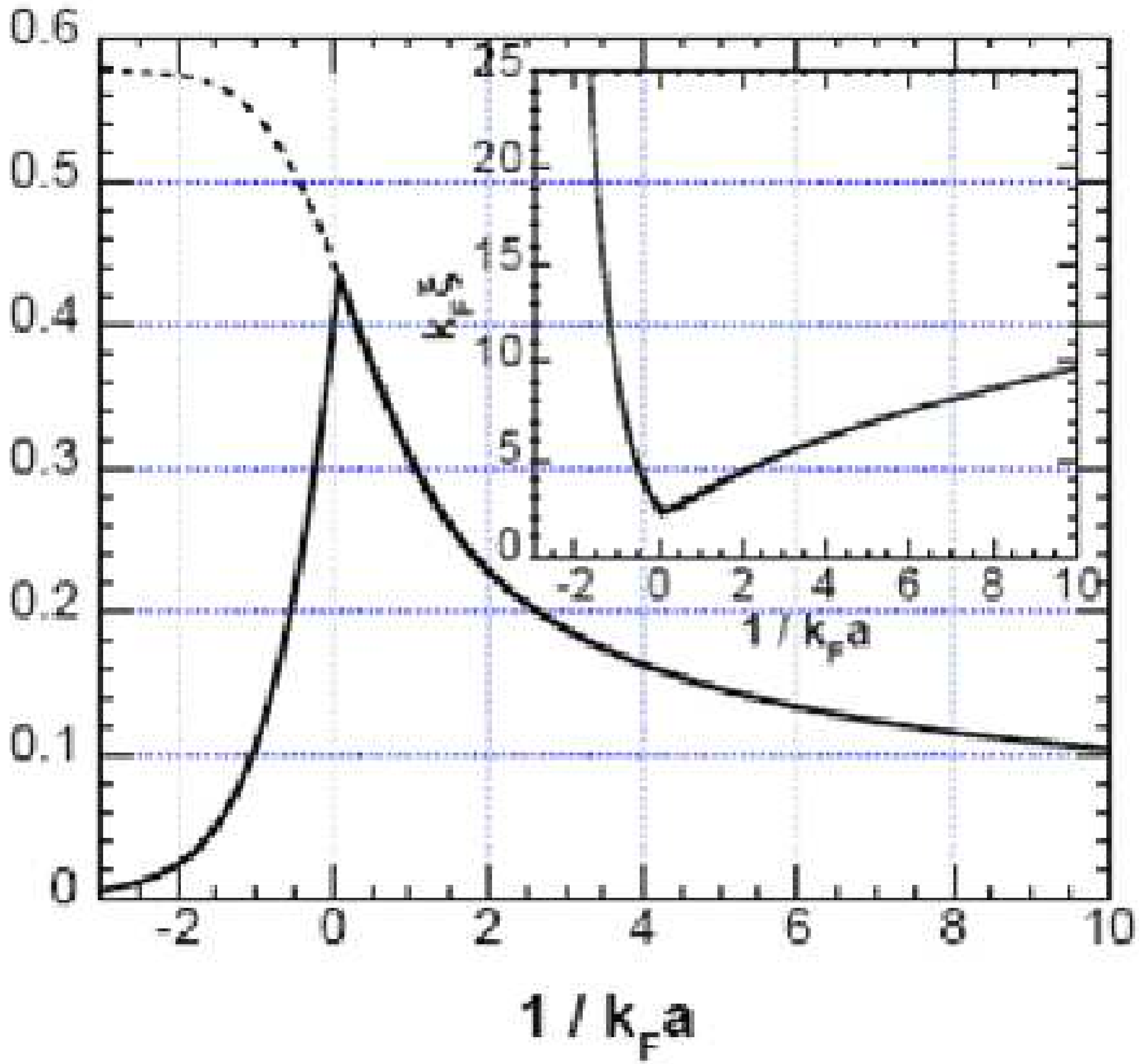
where v_{cr} is the Landau critical velocity (Landau criterion for dissipation).
In the BEC regime of bosonic dimers v_{cr} coincides with the sound velocity, i.e.

$$v_{cr} = c_s = \sqrt{\frac{n}{m}\frac{\partial \mu}{\partial n}} \tag{12}$$

In the BCS regime v_{cr} is instead related to the breaking of Cooper pairs through the formula

$$v_{cr} = \sqrt{\frac{\sqrt{\mu^2 + |\Delta|^2} - \mu}{m}} \tag{13}$$

where $|\Delta|$ is the energy gap of Cooper pairs. Unfortunately, the equations of superfluid hydrodynamics do not take into account the effect of pair breaking.



Landau's critical velocity (in units of Fermi velocity) calculated along the BCS-BEC crossover.

The figure clearly shows that the critical velocity is largest close to unitarity.

From R. Combescot, M.Yu. Kagan and S. Stringari, PRA **74**, 042717 (2006).

Superfluid NLSE for the BCS-BEC crossover

Inspired by Ginzburg-Landau theory, by density functional theory (DFT), and by low-energy effective field theory (EFT), we introduce the complex wave function

$$\Psi(\mathbf{r},t) = \sqrt{\frac{n(\mathbf{r},t)}{2}}\; e^{i\theta(\mathbf{r},t)} \tag{14}$$

which describes boson-like Cooper pairs with the normalization

$$\int |\Psi(\mathbf{r},t)|^2 d^3\mathbf{r} = \frac{N}{2} \tag{15}$$

that is quite different from the normalization of the condensate wave function $\Xi(\mathbf{r},t)$, but the phase $\theta(\mathbf{r},t)$ is the same.

By imposing the (fermionic) velocity-phase relationship

$$\mathbf{v} = \frac{\hbar}{2m}\nabla\theta \tag{16}$$

one can search the simplest nonlinear Schrödinger equation of $\Psi(\mathbf{r},t)$ which satisfies Eq. (16) and reproduces the equations of superfluid hydrodynamics in the classical limit ($\hbar \to 0$).

We find that the nonlinear Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{4m}\nabla^2 + 2U(\mathbf{r}) + 2\mu(n(\mathbf{r},t),a_F)\right]\Psi(\mathbf{r},t) \tag{17}$$

gives the equations of superfluid hydrodynamics in the classical limit ($\hbar \to 0$). In general, our superfluid NLSE gives the hydrodynamics equations with an additional quantum pressure term

$$T_{QP} = -\frac{\hbar^2}{8m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} \tag{18}$$

which depends explicitly on the reduced Planck constant $\hbar$ (gradient correction in DFT, next-to-leading correction in low-energy EFT).

In the deep BEC regime from Eq. (17) one recovers the familiar Gross-Pitaevskii equation for Bose-condensed dimers (molecules of two fermions), where

$$\mu(n,a_F) = \frac{4\pi\hbar^2 a_{dd}(a_F)}{2m} n \tag{19}$$

with $a_{dd}(a_F)$ the dimer-dimer scattering length, which depends on the Fermi-Fermi scattering length a_F. Mean-field theory: $a_{dd}(a_F) = 2a_F$; four-body theory and also MC data: $a_{dd}(a_F) = 0.6a_F$.

Direct current Josephson effect

We use our time-dependent superfluid NLSE

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{4m}\nabla^2 + 2U(\mathbf{r}) + 2\mu(n(\mathbf{r},t),a_F)\right]\Psi(\mathbf{r},t) \tag{20}$$

to study the DC Josephson effect. We choose a square-well barrier

$$U(\mathbf{r}) = \begin{cases} V_0 & \text{for } |z| < d \\ 0 & \text{elsewhere} \end{cases} \tag{21}$$

which separates the superfluid into two regions, and look for a stationary solution

$$\Psi(\mathbf{r},t) = \Phi(\mathbf{r})\ e^{i\theta(\mathbf{r})}\ e^{-i2\bar{\mu}t/\hbar} \tag{22}$$

with <u>constant and uniform supercurrent</u>

$$J = n(\mathbf{r})\mathbf{v}(\mathbf{r}) = 2\Phi(\mathbf{r})^2\frac{\hbar}{2m}\nabla\theta(\mathbf{r}) \tag{23}$$

It follows $(\nabla\theta)^2 = J^2/\Phi^4$ and also

$$\left[-\frac{\hbar^2}{4m}\nabla^2 + \frac{m}{4}\frac{J^2}{\Phi(\mathbf{r})^4} + 2U(\mathbf{r}) + 2\mu(n(\mathbf{r}),a_F)\right]\Phi(\mathbf{r}) = 2\bar{\mu}\ \Phi(\mathbf{r}) \tag{24}$$

We solve the stationary superfluid NLSE by imposing a constant and uniform density $\bar{n}$ at infinity:

$$\Phi(\mathbf{r}) \to \sqrt{\frac{\bar{n}}{2}} \quad \text{for} \quad |\mathbf{r}| \to \infty \tag{25}$$

Given $\Phi(\mathbf{r})$ at fixed J, the phase $\theta(\mathbf{r})$ is then obtained from

$$\theta(\mathbf{r}) = \theta(\mathbf{r}_0) + \frac{mJ}{\hbar} \int_{\mathbf{r}_0}^{\mathbf{r}} \frac{1}{\Phi(\mathbf{r})^2} d\mathbf{r} \tag{26}$$

The phase difference across the barrier is defined as

$$\Delta\theta = \theta(z = +\infty) - \theta(z = -\infty) \tag{27}$$

In this way we are able to determine the relationship between the current J and the phase difference $\Delta\theta$.

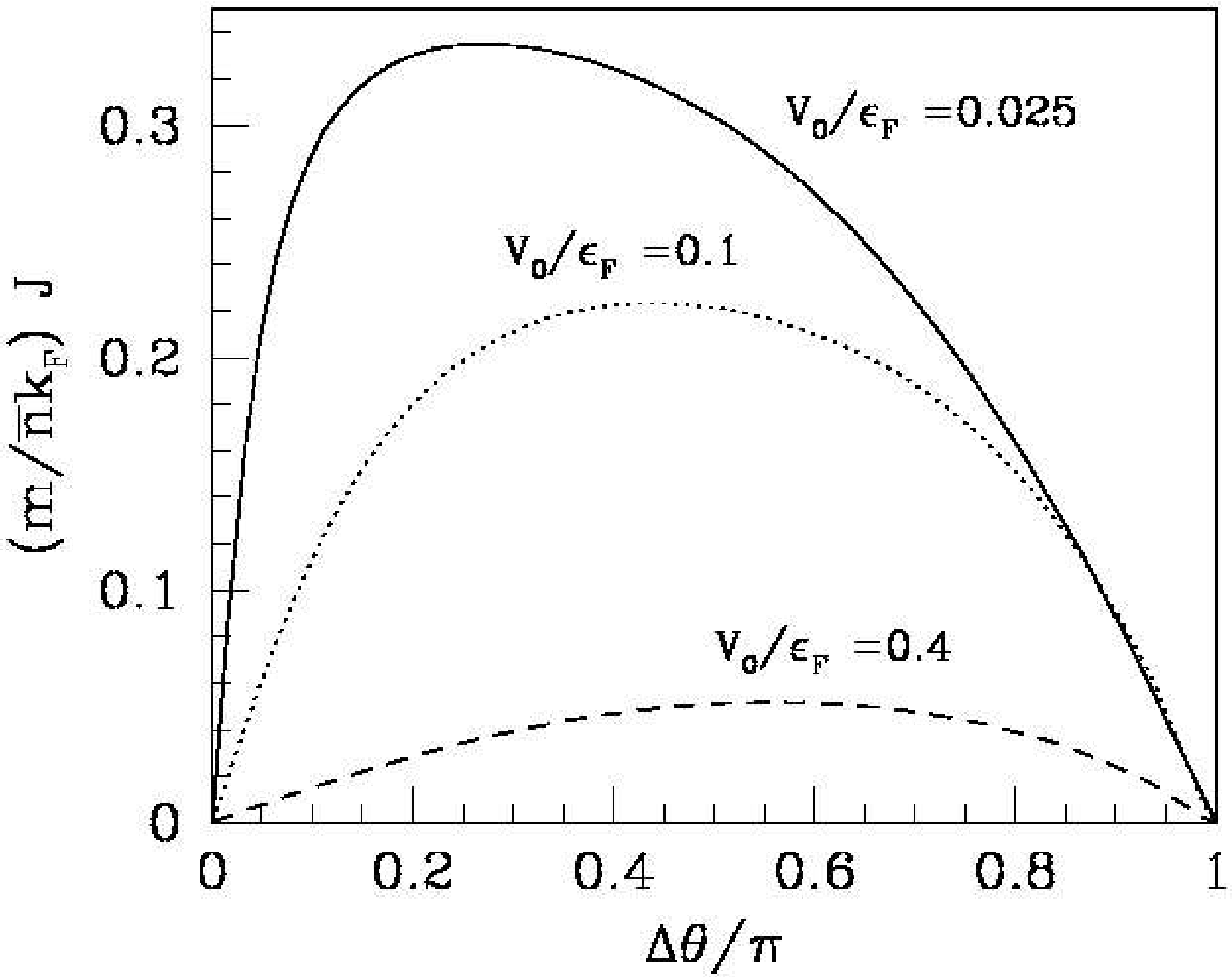
We expect to recover the Josephson equation

$$J = J_0 \sin(\Delta\theta) \tag{28}$$

in the regime of high barrier (small tunneling, weak-link). Instead, in the limit of very small barrier (quasi-free transport, strong-link) J_0 has its maximum value

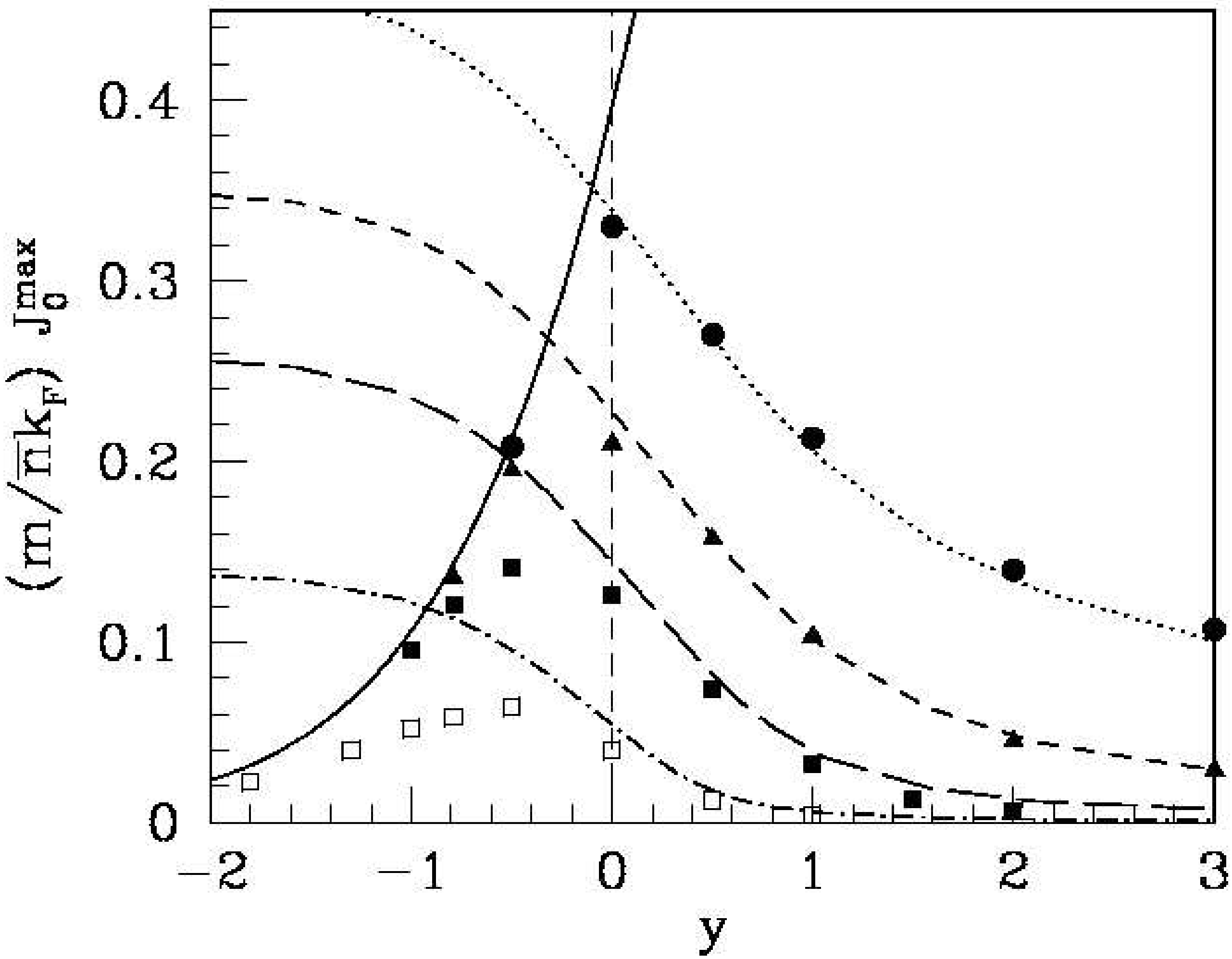
$$J_0^{max} = \bar{n}\ v_{cr} \tag{29}$$

where v_{cr} is the Landau critical velocity.



DC Josephson current J vs phase difference $\Delta\theta$ for a superfluid Fermi gas at unitarity ($y = 0$). Three values of the energy barrier V_0, with $\epsilon_F = \hbar^2(3\pi^2\bar{n})^{2/3}/(2m)$ the Fermi energy. Width of the barrier: $Lk_F = 4$.

From: preliminary results of F. Ancilotto, L.S., F. Toigo (June 2008).



Maximal Josephson current J_0^{max} vs inverse interaction parameter $y = 1/(k_F a_F)$ in the BCS-BEC crossover. Curves: superfluid NLSE. Symbols: microscopic mean-field calculations of Spuntarelli, Pieri, Strinati [PRL **99**, 040401 (2007)]. Four values of the energy barrier V_0/ϵ_F: 0.025, 0.10, 0.2, 0.4. Width of the barrier: $Lk_F = 4$.

From: preliminary results of F. Ancilotto, L.S., F. Toigo (June 2008).

Alternate current Josephson effect

Let us now consider a high barrier (small tunneling, weak-link) without imposing a constant supercurrent J.

We start from our time-dependent superfluid NLSE

$$i\hbar \frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{4m}\nabla^2 + 2U(\mathbf{r}) + 2\mu(n(\mathbf{r},t),a_F) \right] \Psi(\mathbf{r},t) \tag{30}$$

and look for a time-dependent solution of the form

$$\Psi(\mathbf{r},t) = c_A(t)\ \Phi_A(\mathbf{r}) + c_B(t)\ \Phi_B(\mathbf{r}) \tag{31}$$

where $\Phi_A(\mathbf{r})$ and $\Phi_B(\mathbf{r})$ is the quasi-stationary solutions normalized to one and localized in region A and B respectively.
In this way we obtain* the following two-state model

$$i\hbar \frac{\partial}{\partial t} c_A(t) = E_A(t)\ c_A(t) + K\ c_B(t) \tag{32}$$

$$i\hbar \frac{\partial}{\partial t} c_B(t) = E_B(t)\ c_B(t) + K\ c_A(t) \tag{33}$$

for the two complex coefficients $c_A(t)$ and $c_B(t)$, which give the number of atoms in the two regions.

*L.S., N. Manini, F. Toigo, PRA **77**, 043609 (2008).

In our two-state model, $E_A(t)$ is the time-dependent energy in region A, given by

$$E_A(t) \simeq \int \Phi_A(\mathbf{r}) \left[-\frac{\hbar^2}{4m}\nabla^2 + 2U(\mathbf{r}) + 2\mu\left(2|c_A(t)|^2\Phi_A(\mathbf{r})^2, a_F\right)\right] \Phi_A(\mathbf{r})\, d^3\mathbf{r} \tag{34}$$

There is obviously a similar expression for the time-dependent energy $E_B(t)$.

The constant coupling energy K describes instead the tunneling between the two regions A and B:

$$K \simeq \int \Phi_A(\mathbf{r}) \left[-\frac{\hbar^2}{4m}\nabla^2 + 2U(\mathbf{r})\right] \Phi_B(\mathbf{r})\, d^3\mathbf{r} \tag{35}$$

From our previous analysis of the DC Josephson effect, we expect that this expression is correct only in the right side of the BCS-BEC crossover.

To study the Josephson effect in the left side of the BCS-BEC crossover one can use K as a phenomenological parameter à la Feynman.[†]

[†]R.P. Feynman, R. Leighton, M. Sands, *Feynman Lectures on Physics*, vol. 3 (Addison Wesley, Reading, 1966);
R.P. Feynman, *Statistical Mechanics. A set of Lectures* (Benjamin, Reading, 1973).

We can write the complex coefficient $c_A(t)$ as

$$c_A(t) = \sqrt{\frac{N_A(t)}{2}}\; e^{i\theta_A(t)} \tag{36}$$

with $N_A(t)$ number of atoms and $\theta(t)$ phase in region A. Again, there is a similar expression for $c_B(t)$.
In terms of the phase difference

$$\varphi(t) = \theta_B(t) - \theta_A(t) \tag{37}$$

and relative number imbalance

$$z(t) = \frac{N_B(t) - N_A(t)}{N_A(t) + N_B(t)} = \frac{N_B(t) - N_A(t)}{N} \,, \tag{38}$$

the two-mode equations give

$$\begin{aligned} \dot{z}(t) &= -\frac{2K}{\hbar}\sqrt{1 - z(t)^2}\, \sin\varphi(t) \,, \\ \dot{\varphi}(t) &= \frac{2}{\hbar}\left[\mu\left(\frac{N}{2V}(1 + z(t))\right) - \mu\left(\frac{N}{2V}(1 - z(t))\right)\right] + \frac{2K}{\hbar}\frac{z(t)}{\sqrt{1 - z(t)^2}}\cos\varphi(t) \end{aligned}$$

These are the atomic Josephson junction (AJJ) equations describing the oscillations of N Fermi atoms tunneling in the superfluid state between region A and region B, both with volume V.

Our nonlinear AJJ equations can be linearized around the stable stationary solution

$$\bar{z} = 0 \quad \text{and} \quad \bar{\varphi} = 2\pi j \tag{39}$$

where j is an integer. One finds the following frequency of small oscillation

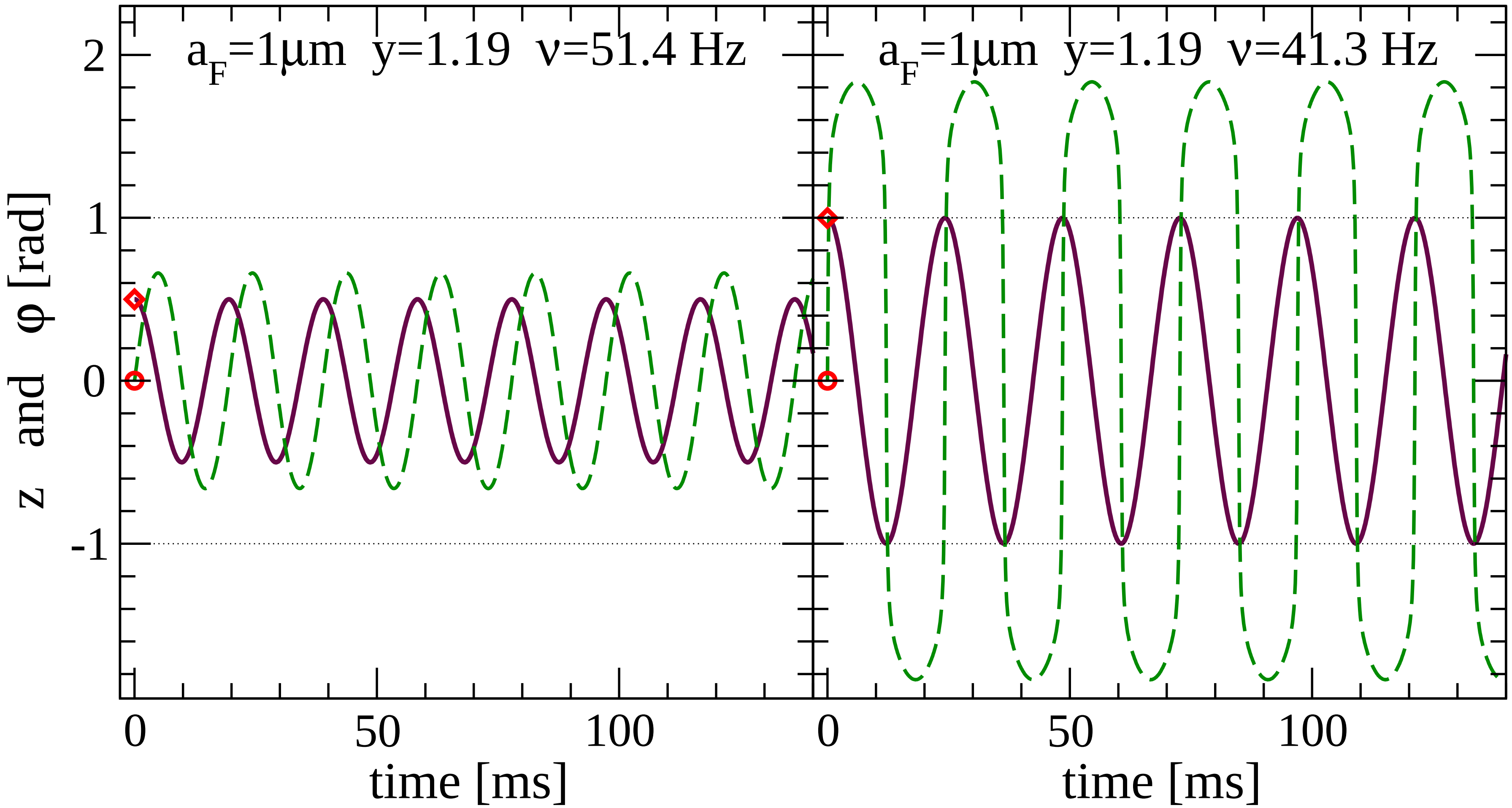
$$\nu_0 = \frac{K}{\pi\hbar}\sqrt{1 + \frac{2mc_s^2}{K}} \tag{40}$$

which is called zero-mode. Here c_s is the sound velocity computed at the mean density $n = N/V$ of the superfluid. The zero-mode is the analog of the Josephson plasma frequency in superconducting junctions.

From our AJJ equations[‡] one finds also the π-mode solution with $\bar{z} = 0$ and $\bar{\varphi} = \pi(2j+1)$ and the self-trapping solution with population imbalance $(\bar{z} \neq 0)$.

Notice that our AJJ equations generalize the BJJ equations obtained by Smerzi *et al.* [PRL **79**, 4950 (1997)] for Bose-Einstein condensates.

[‡]L.S., N. Manini, F. Toigo, PRA **77**, 043609 (2008).



Zero-mode in the AC Josephson effect by solving AJJ equations. $N = 10^6$ ^{40}K atoms between two symmetric regions of volume $25 \cdot 10^6$ μm^3, tunneling parameter $K/k_{\rm B} = 10^{-9}$ Kelvin and Fermi-Femri scattering length $a_F = 1$ μm, corresponding to $y = 1.19$. Solid line: population imbalance $z(t)$; dashed line: phase difference $\varphi(t)$. Initial conditions: $\varphi(0) = 0$ and $z(0) = 0.5$ (left); $\varphi(0) = 0$ and $z(0) = 0.999$ (right).

From L.S., N. Manini, F. Toigo, PRA **77**, 043609 (2008).

Conclusions

- We have discussed a superfluid NLSE which gives hydrodynamic equations of Fermi superfluids plus a gradient correction.

- Hydrodynamics equations and superfluid NLSE are reliable to investigate:
  i) static properties;
  ii) low-energy collective dynamics.

- In the study of the DC Josephson effect, our superfluid NLSE works quite well in the right side of the BCS-BEC crossover.

- In the study of DC and AC Josephson effects, our superfluid NLSE works efficiently for the full BCS-BEC crossover by using a phemenological tunneling parameter in the left side of the crossover. In this way we get the AJJ equations.

THANKS!!