Superfluidity in 2D systems: BCS-BEC crossover and BEC on the surface of a sphere

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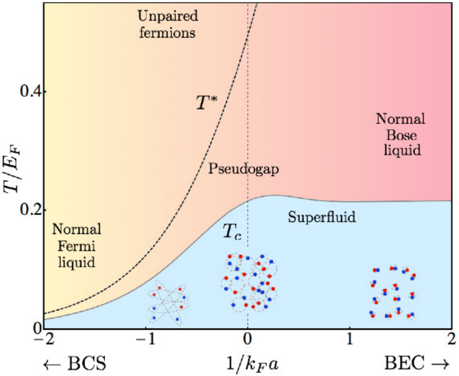
A1. BCS-BEC crossover in 3D and 2D (I)

In 2004 the 3D BCS-BEC crossover has been observed with **ultracold gases made of two-component fermionic ^{40}K or ^{6}Li atoms.**[1]



This crossover is obtained using a Fano-Feshbach resonance to change the 3D s-wave scattering length a_F of the inter-atomic potential.

[1] C.A. Regal et al., PRL **92**, 040403 (2004); M.W. Zwierlein et al., PRL **92**, 120403 (2004); J. Kinast et al., PRL **92**, 150402 (2004).

A1. BCS-BEC crossover in 3D and 2D (II)

Recently also the 2D BEC-BEC crossover has been achieved experimentally[2] with a **Fermi gas of two-component ^{6}Li atoms**. In 2D attractive fermions always form biatomic molecules with bound-state energy

$$\epsilon_B \simeq \frac{\hbar^2}{m{a_F}^2} \, , \tag{1}$$

where a_F is the 2D s-wave scattering length, which is experimentally tuned by a Fano-Feshbach resonance.
The fermionic single-particle spectrum is given by

$$E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2} \, , \tag{2}$$

where Δ_0 is the energy gap and μ is the chemical potential: $\mu > 0$ corresponds to the BCS regime while $\mu < 0$ corresponds to the BEC regime. Moreover, in the deep BEC regime $\mu \to -\epsilon_B/2$.

[2] V. Makhalov et al. PRL **112**, 045301 (2014); M.G. Ries et al., PRL **114**, 230401 (2015); I. Boettcher et al., PRL **116**, 045303 (2016); K. Fenech et al., PRL **116**, 045302 (2016).

A2. 2D equation of state (I)

To study the 2D BCS-BEC crossover we adopt the formalism of functional integration[3]. The partition function $\mathcal{Z}$ of the uniform system with fermionic fields $\psi_s(\mathbf{r},\tau)$ at temperature T, in a 2-dimensional volume L^2, and with chemical potential μ reads

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \ \exp\left\{ -\frac{S}{\hbar} \right\} , \tag{3}$$

where ($\beta \equiv 1/(k_B T)$ with k_B Boltzmann's constant)

$$S = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2\mathbf{r} \ \mathcal{L} \tag{4}$$

is the Euclidean action functional with Lagrangian density

$$\mathcal{L} = \bar{\psi}_s \left[\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \psi_s + \mathbf{g}\, \bar{\psi}_\uparrow \, \bar{\psi}_\downarrow \, \psi_\downarrow \, \psi_\uparrow \tag{5}$$

where $\mathbf{g}$ is the attractive strength ($\mathbf{g} < 0$) of the s-wave coupling.

[3] N. Nagaosa, Quantum Field Theory in Condensed Matter (Springer, 1999).

A2. 2D equation of state (II)

Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathcal{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(\mathbf{r}, \tau)$. In this way the effective Euclidean Lagrangian density reads

$$\mathcal{L}_e = \bar{\psi}_s \left[\hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + \bar{\Delta} \, \psi_\downarrow \, \psi_\uparrow + \Delta \bar{\psi}_\uparrow \, \bar{\psi}_\downarrow - \frac{|\Delta|^2}{\mathbf{g}} \, . \qquad (6)$$

We investigate the effect of fluctuations of the pairing field $\Delta(\mathbf{r}, t)$ around its mean-field value Δ_0 which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r}, \tau) = \Delta_0 + \eta(\mathbf{r}, \tau) \, , \qquad (7)$$

where $\eta(\mathbf{r}, \tau)$ is the complex field which describes pairing fluctuations.

A2. 2D equation of state (III)

In particular, we are interested in the grand potential Ω, given by

$$\Omega = -\frac{1}{\beta} \ln (\mathcal{Z}) \simeq -\frac{1}{\beta} \ln (\mathcal{Z}_{mf} \mathcal{Z}_g) = \Omega_{mf} + \Omega_g \ , \tag{8}$$

where

$$\mathcal{Z}_{mf} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar} \right\} \tag{9}$$

is the mean-field partition function and

$$\mathcal{Z}_g = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \mathcal{D}[\eta, \bar{\eta}] \ \exp \left\{ -\frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar} \right\} \tag{10}$$

is the partition function of Gaussian pairing fluctuations.

A2. 2D equation of state (IV)

After functional integration over quadratic fields, one finds that the mean-field grand potential reads[4]

$$\Omega_{mf} = -\frac{\Delta_0^2}{\mathbf{g}} L^2 + \sum_{\mathbf{k}} \left(\frac{\hbar^2 k^2}{2m} - \mu - E_{sp}(\mathbf{k}) - \frac{2}{\beta} \ln\left(1 + e^{-\beta E_{sp}(\mathbf{k})}\right) \right) \quad (11)$$

where

$$E_{sp}(\mathbf{k}) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu \right)^2 + \Delta_0^2} \quad (12)$$

is the spectrum of fermionic single-particle excitations.

[4] A. Altland and B. Simons, Condensed Matter Field Theory (Cambridge Univ. Press, 2006).

A2. 2D equation of state (V)

The Gaussian grand potential is instead given by

$$\Omega_g = \frac{1}{2\beta} \sum_Q \ln \det(\mathbf{M}(Q)) \; , \tag{13}$$

where $\mathbf{M}(Q)$ is the inverse propagator of Gaussian fluctuations of pairs and $Q = (\mathbf{q}, i\Omega_m)$ is the 4D wavevector with $\Omega_m = 2\pi m/\beta$ the Matsubara frequencies and $\mathbf{q}$ the 3D wavevector.[5]

The sum over Matsubara frequencies is quite complicated and it does not give a simple expression. An approximate formula[6] is

$$\Omega_g \simeq \frac{1}{2} \sum_{\mathbf{q}} E_{col}(\mathbf{q}) + \frac{1}{\beta} \sum_{\mathbf{q}} \ln\left(1 - e^{-\beta\, E_{col}(\mathbf{q})}\right) \; , \tag{14}$$

where

$$E_{col}(\mathbf{q}) = \hbar\, \omega(\mathbf{q}) \tag{15}$$

is the spectrum of bosonic collective excitations with $\omega(\mathbf{q})$ derived from

$$\det(\mathbf{M}(\mathbf{q}, \omega)) = 0 \; . \tag{16}$$

[5] R.B. Diener, R. Sensarma, M. Randeria, PRA **77**, 023626 (2008).
[6] E. Taylor, A. Griffin, N. Fukushima, Y. Ohashi, PRA **74**, 063626 (2006).

A2. 2D equation of state (VI)

In our approach (Gaussian pair fluctuation theory[7]), given the grand potential

$$\Omega(\mu, L^2, T, \Delta_0) = \Omega_{mf}(\mu, L^2, T, \Delta_0) + \Omega_g(\mu, L^2, T, \Delta_0) \ , \tag{17}$$

the energy gap Δ_0 is obtained from the (mean-field) gap equation

$$\frac{\partial \Omega_{mf}(\mu, L^2, T, \Delta_0)}{\partial \Delta_0} = 0 \ . \tag{18}$$

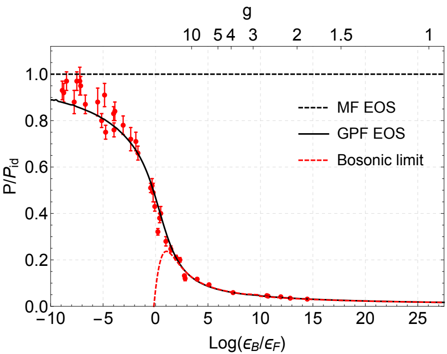
The number density n is instead obtained from the number equation

$$n = -\frac{1}{L^2} \frac{\partial \Omega(\mu, L^2, T, \Delta_0(\mu, T))}{\partial \mu} \tag{19}$$

taking into account the gap equation, i.e. that Δ_0 depends on μ and T: $\Delta_0(\mu, T)$. Notice that the Nozieres and Schmitt-Rink approach[8] is quite similar but in the number equation it forgets that Δ_0 depends on μ.

[7] H. Hu, X-J. Liu, P.D. Drummond, EPL **74**, 574 (2006).
[8] P. Nozieres and S. Schmitt-Rink, JLTP **59**, 195 (1985).

A3. Zero-temperature 2D results (I)



Scaled pressure P/P_{id} vs scaled binding energy ϵ_B/ϵ_F. Notice that $P = -\Omega/L^2$ and P_{id} is the pressure of the ideal 2D Fermi gas. Filled squares with error bars: experimental data of Makhalov *et al.* [9]. Solid line: the regularized Gaussian theory[10].

[9] V. Makhalov et al. PRL **112**, 045301 (2014).

[10] G. Bighin and LS, PRB **93**, 014519 (2016). See also L. He, H. Lu, G. Cao, H. Hu and X.-J. Liu, PRA **92**, 023620 (2015).

A3. Zero-temperature 2D results (II)

In the analysis of the **two-dimensional attractive Fermi gas** one must remember that, contrary to the 3D case, 2D realistic interatomic attractive potentials have always a bound state. In particular[11], the binding energy $\epsilon_B > 0$ of two fermions can be written in terms of the positive 2D fermionic scattering length a_F as

$$\epsilon_B = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{m {a_F}^2} \; , \tag{20}$$

where $\gamma = 0.577...$ is the Euler-Mascheroni constant. Moreover, the attractive (negative) interaction strength **g** of s-wave pairing is related to the binding energy $\epsilon_B > 0$ of a fermion pair in vacuum by the expression[12]

$$-\frac{1}{\mathbf{g}} = \frac{1}{2L^2} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 k^2}{2m} + \frac{1}{2}\epsilon_B} \; . \tag{21}$$

[11] C. Mora and Y. Castin, 2003, PRA **67**, 053615.
[12] M. Randeria, J-M. Duan, and L-Y. Shieh, PRL **62**, 981 (1989).

A3. Zero-temperature 2D results (III)

At zero temperature, including Gaussian fluctuations, the pressure is

$$P = -\frac{\Omega}{L^2} = \frac{mL^2}{2\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2 + P_g(\mu, L^2, T = 0) \ , \qquad (22)$$

with

$$P_g(\mu, L^2, T = 0) = -\frac{1}{2}\sum_{\mathbf{q}} E_{col}(\mathbf{q}) \ . \qquad (23)$$

In the full 2D BCS-BEC crossover, from the regularized version of Eq. (13), we obtain numerically the zero-temperature pressure[13]
Notice that the energy of bosonic collective excitations becomes

$$E_{col}(\mathbf{q}) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda\frac{\hbar^2 q^2}{2m} + 2mc_s^2\right)} \qquad (24)$$

in the deep BEC regime, with $\lambda = 1/4$ and $mc_s^2 = \mu + \epsilon_B/2$.

[13] G. Bighin and LS, PRB **93**, 014519 (2016). See also L. He, H. Lu, G. Cao, H. Hu and X.-J. Liu, PRA **92**, 023620 (2015).

A3. Zero-temperature 2D results (IV)

In the deep BEC regime of the **2D BCS-BEC crossover**, where the chemical potential μ becomes strongly negative, the corresponding regularized pressure (dimensional regularization [14]) reads

$$P = \frac{m}{64\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2 \; \ln\left(\frac{\epsilon_B}{2(\mu + \frac{1}{2}\epsilon_B)}\right) . \qquad (25)$$

This is exactly the Popov equation of state of 2D Bose gas with chemical potential $\mu_B = 2(\mu + \epsilon_B/2)$, mass $m_B = 2m$. In this way we have identified the two-dimensional scattering length a_B of composite boson as

$$\boxed{a_B = \frac{1}{2^{1/2}e^{1/4}} \; a_F \; .} \qquad (26)$$

The value $a_B/a_F = 1/(2^{1/2}e^{1/4}) \simeq 0.551$ is in full agreement with $a_B/a_F = 0.55(4)$ obtained by Monte Carlo calculations[15].

[14] LS and F. Toigo, PRA **91**, 011604(R) (2015); LS, PRL **118**, 130402 (2017).
[15] G. Bertaina and S. Giorgini, PRL **106**, 110403 (2011).

A4. Finite-temperature 2D results (I)

We are now interested on the temperature dependence of superfluidy density $n_s(T)$ of the system.
At the Gaussian level $n_s(T)$ depends only on fermionic single-particle excitations $E_{sp}(k)$.[16] **Beyond the Gaussian level** also bosonic collective excitations $E_{col}(q)$ contribute.[17]
Thus, we assume the following Landau-type formula for the superfluid density[18]

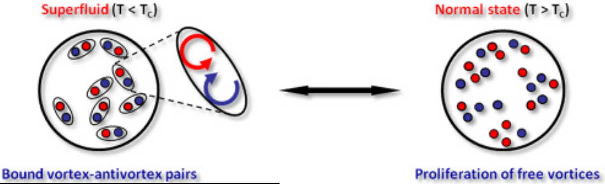
$$n_s(T) = n - \beta \int \frac{\mathrm{d}^2 k}{(2\pi)^2} k^2 \frac{e^{\beta E_{sp}(k)}}{(e^{\beta E_{sp}(k)}+1)^2} - \frac{\beta}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} q^2 \frac{e^{\beta E_{col}(q)}}{(e^{\beta E_{col}(q)}-1)^2} . \tag{27}$$

[16] E. Babaev and H.K. Kleinert, PRB **59**, 12083 (1999).
[17] L. Benfatto, A. Toschi, and S. Caprara, PRB **69**, 184510 (2004).
[18] G. Bighin and LS, PRB **93**, 014519 (2016).

A4. Finite-temperature 2D results (IV)

The analysis of **Kosterlitz** and **Thouless**[19] applied to 2D superfluids shows that:

- As the temperature T increases vortices start to appear in vortex-antivortex pairs.
- The pairs are bound at low temperature until at the critical temperature $T_c = T_{BKT}$ an unbinding transition occurs above which a proliferation of free vortices and antivortices is predicted.
- The superfluid density $n_s(T)$ is renormalized by the presence of vortex-antivortex pairs.
- The renormalized superfluid density $n_{s,R}(T)$ decreases by increasing the temperature T and jumps to zero at $T_c = T_{BKT}$.



[19] J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973).

A4. Finite-temperature 2D results (V)

We have seen that the renormalized superfluid density $n_{s,R}(T)$ jumps to zero at a critical temperature T_{BKT}.
Moreover, one finds the Nelson-Kosterlitz condition[20]

$$k_B T_{BKT} = \frac{\hbar^2 \pi}{8m} n_{s,R}(T_{BKT}^-) \ . \tag{28}$$

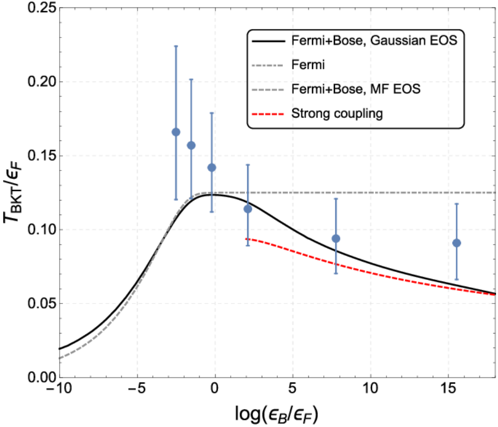
Often the following Nelson-Kosterlitz criterion is adopted

$$k_B T_{BKT} = \frac{\hbar^2 \pi}{8m} n_s(T_{BKT}) \ , \tag{29}$$

with $n_s(T)$ **instead of** $n_{s,R}(T)$. In this way one gets an approximated[21] T_{BKT} without the effort of calculating the renormalized superfluid density $n_{s,R}(T)$.

[20] D.R. Nelson and J.M. Kosterlitz, Phys Rev. Lett. **39**, 1201 (1977).
[21] An improved approach based on the RG equations of Kosterlitz and Thouless can be found in G. Bighin and LS, Sci. Rep. **7**, 45702 (2017).

A4. Finite-temperature 2D results (VI)



Our theoretical predictions[22] for the Berezinskii-Kosterlitz-Thouless critical temperature T_{BKT} compared to experimental observation[23] (filled circles with error bars).

[22] G. Bighin and LS, PRB **93**, 014519 (2016).
[23] P.A. Murthy et al., PRL **115**, 010401 (2015).

B1. Bose gas on the surface of a sphere (I)

Recently, Bose-Einstein condensates (BECs) made of ultracold alkali-metal atoms under microgravity have been achieved
i) dropping the BEC down a 146-meter-long drop chamber[24]
ii) rocketing the BEC and conducting experiments during in-space flight[25]

In addition, in 2018 a NASA's Cold Atom Laboratory (CAL) was successfully launched aboard an Orbital ATK Cygnus spacecraft.[26] In the near future, CAL will operate in the microgravity environment of the International Space Station.

[24] T. van Zoest, et al., Science **328**, 1540 (2010)
[25] D. Becker et al., Nature 562, 391 (2018).
[26] See the webpage https://coldatomlab.jpl.nasa.gov

B1. Bose gas on the surface of a sphere (II)

Our theoretical study of a Bose gas on the surface of a sphere is triggered by the experimental possibility to confine the atoms on a bubble trap,[27] which needs microgravity conditions.[28]
The energy of a particle of mass m moving on the surface of a sphere of radius R is quantized according to the formula

$$\epsilon_l = \frac{\hbar^2}{2mR^2} l(l+1) \; , \tag{30}$$

where $\hbar$ is the reduced Planck constant and $l = 0, 1, 2, ...$ is the **integer quantum number** of the angular momentum. This energy level has the degeneracy $2l + 1$ due to the magnetic quantum number $m_l = -l, -l+1, ..., l-1, l$ of the third component of the angular momentum.

[27] B. M. Garraway and H. Perrin, J. Phys. B **49**, 172001 (2016).
[28] E.R. Elliott et al., npj Microgravity **4**, 16 (2018).

B2. Non-interacting bosons: critical temperature (I)

In quantum statistical mechanics the total number N of non-interacting bosons moving on the surface of a sphere and at equilibrium with a thermal bath of absolute temperature T is given by

$$N = \sum_{l=0}^{+\infty} \frac{2l+1}{e^{(\epsilon_l - \mu)/(k_B T)} - 1} \ , \qquad (31)$$

where k_B is the Boltzmann constant and μ is the chemical potential. In the Bose-condensed phase, we can set[29] $\mu = 0$ and

$$N = N_0 + \sum_{l=1}^{+\infty} \frac{2l+1}{e^{\epsilon_l/(k_B T)} - 1} \ , \qquad (32)$$

where N_0 is the number of bosons in the lowest single-particle energy state, i.e. the number of bosons in the Bose-Einstein condensate (BEC).

[29] For details, see Martina Russo, BSc thesis, Supervisor: LS, Univ. of Padova (2019).

B2. Non-interacting bosons: critical temperature (II)

Within the semiclassical approximation, where $\sum_{l=1}^{+\infty} \to \int_1^{+\infty} dl$, the previous equation becomes

$$n = n_0 + \frac{mk_B T}{2\pi\hbar^2}\left(\frac{\hbar^2}{mR^2 k_B T} - \ln\left(e^{\hbar^2/(mR^2 k_B T)} - 1\right)\right) , \qquad (33)$$

where $n = N/(4\pi R^2)$ is the 2D number density and $n_0 = N_0/(4\pi R^2)$ is the 2D condensate density.

At the critical temperature T_{BEC}, where $n_0 = 0$, one then finds[30]

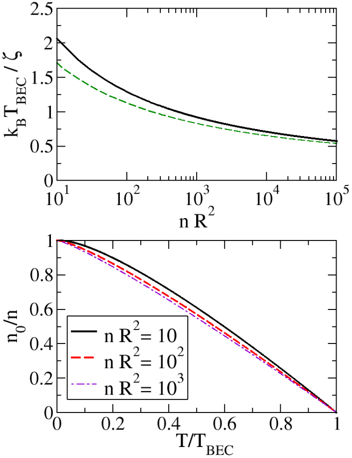
$$k_B T_{BEC} = \frac{\frac{2\pi\hbar^2}{m} n}{\frac{\hbar^2}{mR^2 k_B T_{BEC}} - \ln\left(e^{\hbar^2/(mR^2 k_B T_{BEC})} - 1\right)} . \qquad (34)$$

As expected, in the limit $R \to +\infty$ one gets $T_{BEC} \to 0$, in agreement with the Mermin-Wagner theorem.[31] However, for any finite value of R the critical temperature T_{BEC} is larger than zero.

[30] A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).

[31] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

B2. Non-interacting bosons: critical temperature (III)



Top panel: T_{BEC} vs nR^2, with $\zeta = \hbar^2 n/m$. Solid line: semiclassical approximation (solid line); dashed line: numerical evaluation of the sum. **Bottom panel**: condensate fraction n_0/n vs temperature T/T_{BEC}.

B3. Interacting bosons: phase diagram (I)

We now consider a system of interacting bosons on the surface of a sphere of radius R and contact interaction of strength g.
Within the formalism of functional integration, the grand canonical partition function reads

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}, \psi] \, e^{-\frac{S[\bar{\psi},\psi]}{\hbar}}, \tag{35}$$

where, by using $\beta = 1/(k_B T)$ with T the absolute temperature,

$$S[\bar{\psi}, \psi] = \int_0^{\beta\hbar} d\tau \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\theta) \, d\theta \, R^2 \, \mathcal{L}(\bar{\psi}, \psi) \tag{36}$$

is the Euclidean action and, with $\hat{L}$ is the angular momentum operator,

$$\mathcal{L} = \bar{\psi}(\theta, \varphi, \tau)\left(\hbar\partial_\tau + \frac{\hat{L}^2}{2mR^2} - \mu\right)\psi(\theta, \varphi, \tau) + \frac{g}{2}|\psi(\theta, \varphi, \tau)|^4 \tag{37}$$

is the Euclidean Lagrangian of the bosonic field $\psi(\theta, \phi, \tau)$, which depends on the spherical angles θ and ϕ and on the imaginary time τ.

B3. Interacting bosons: phase diagram (II)

The condensate phase is introduced with the Bogoliubov shift

$$\psi(\theta, \varphi, \tau) = \psi_0 + \eta(\theta, \varphi, \tau), \tag{38}$$

where the real field configuration ψ_0 describes the condensate component. By substituting this field parametrization and keeping only second order terms in the field η we rewrite the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g \tag{39}$$

with $\mathcal{L}_0 = -\mu {\psi_0}^2 + g {\psi_0}^4/2$.
We use the following decomposition of the complex fluctuation field $\eta(\theta, \varphi, \tau)$

$$\eta(\theta, \varphi, \tau) = \sum_{\omega_n} \sum_{l=1}^{\infty} \sum_{m_l=-l}^{l} \frac{e^{-i\omega_n \tau}}{R} \mathcal{Y}^l_{m_l}(\theta, \varphi)\, \eta(l, m_l, \omega_n), \tag{40}$$

where $\omega_n = 2\pi n/(\hbar\beta)$ are the Matsubara frequencies, and we introduce the orthonormal basis of the spherical harmonics $\mathcal{Y}^l_{m_l}(\theta, \phi)$.

B3. Interacting bosons: phase diagram (III)

After some analytical calculations, at the Gaussian level the grand potential

$$\Omega = -\frac{1}{\beta}\ln(\mathcal{Z}) \simeq -\frac{1}{\beta}\left(\ln(\mathcal{Z}_0) + \ln(\mathcal{Z}_g)\right) \quad (41)$$

is given by

$$\begin{aligned}\Omega(\mu, {\psi_0}^2) &= 4\pi R^2\left(-\mu{\psi_0}^2 + g{\psi_0}^4/2\right) + \frac{1}{2}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} E_l(\mu, {\psi_0}^2) \\ &+ \frac{1}{\beta}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l} \ln(1 - e^{-\beta E_l(\mu, {\psi_0}^2)}) \quad (42)\end{aligned}$$

where

$$E_l(\mu, {\psi_0}^2) = \sqrt{(\epsilon_l - \mu + 2g{\psi_0}^2)^2 - g^2{\psi_0}^4} \quad (43)$$

is the excitation spectrum of the interacting system, with $\epsilon_l = \hbar^2 l(l+1)/(2mR^2)$ the single-particle energy.

B3. Interacting bosons: phase diagram (IV)

The condensate number density n_0 of the system is given by

$$n_0 = {\psi_0}^2 \ , \tag{44}$$

where we fix the value of the order parameter ψ_0 with the condition

$$\frac{\partial \Omega(\mu, {\psi_0}^2)}{\partial \psi_0} = 0 \ . \tag{45}$$

Notice that from this formula we get n_0 as a function of μ.
The total number density of the system is instead given by

$$n = -\frac{1}{4\pi R^2} \frac{\partial \Omega(\mu, n_0(\mu))}{\partial \mu} \ . \tag{46}$$

At the lowest order of a perturbative scheme,[32] where ψ_0 is obtained from the mean-field equation $\frac{\partial \Omega_0(\mu, {\psi_0}^2)}{\partial \psi_0} = 0$, we get $\psi_0 \simeq \sqrt{\mu/g}$ and

$$E_l \simeq E_l^B = \sqrt{\epsilon_l(\epsilon_l + 2\mu)} \ . \tag{47}$$

[32] H. Kleinert, S. Schmidt, and A. Pelster, Phys. Rev. Lett. **93**, 160402 (2004).

B3. Interacting bosons: phase diagram (V)

Within this perturbative scheme[33] from the previous equations we obtain[34] the BEC critical temperature

$$k_B T_{BEC} = \frac{\frac{2\pi\hbar^2 n}{m} - \frac{gn}{2}}{\frac{\hbar^2}{2mR^2 k_B T_{BEC}}\left(1+\sqrt{1+\frac{2gmnR^2}{\hbar^2}}\right) - \ln\left(e^{\frac{\hbar^2}{mR^2 k_B T_{BEC}}\sqrt{1+\frac{2gmnR^2}{\hbar^2}}} - 1\right)}, \quad (48)$$

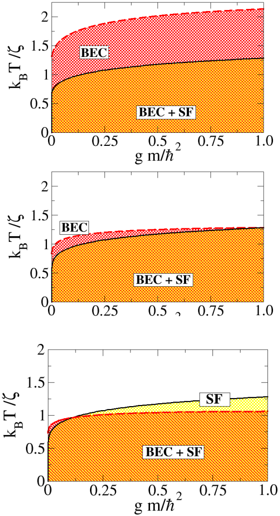
where the condensate density n_0 is zero.
Moreover, adopting the Landau formula for the normal density, we calculate the superfluid density $n_s(T)$ as

$$n_s = n - \frac{1}{k_B T}\int_1^{+\infty} \frac{dl\,(2l+1)}{4\pi R^2}\,\frac{\hbar^2(l^2+l)}{2mR^2}\,\frac{e^{E_l^B/(k_B T)}}{(e^{E_l^B/(k_B T)}-1)^2}, \quad (49)$$

and applying the Kosterlitz-Nelson criterion we evaluate numerically the BKT critical temperature T_{BKT}.

[33] H. Kleinert, S. Schmidt, and A. Pelster Phys. Rev. Lett. **93**, 160402 (2004).
[34] A. Tononi and LS, Phys. Rev. Lett. **123**, 160403 (2019).

B3. Interacting bosons: phase diagram (VI)



Phase diagram of the bosonic system for $nR^2 = 10^2$ (**upper panel**), $nR^2 = 10^4$ (**middle panel**), $nR^2 = 10^5$ (**lower panel**). Here $\zeta = \hbar^2 n/m$.

Conclusions

- We have found that in the **2D BCS-BEC crossover**, after regularization[35] beyond-mean-field Gaussian fluctuations and quantized vortices give remarkable effects for superfluid fermions:
  – logarithmic behavior of the equation of state in the deep BEC regime
  – good agreement with (quasi) zero-temperature experimental data
  – bare n_s and renormalized $n_{s,R}$ superfluid density
  – Berezinskii-Kosterlitz-Thouless critical temperature T_{BKT}
- Triggered by recent achievements of space-based BECs under microgravity and bubble traps, which confine atoms on a thin shell, we have investigated **BEC on the surface of a sphere** finding:
  – BEC critical temperature for non-interacting bosons
  – BEC and BKT critical temperatures for interacting bosons
- Finite-range effects of the inter-atomic potential could be included within an effective-field-theory (EFT) approach.[36]

[35] For a recent comprehensive review see LS and F. Toigo, Phys. Rep. **640**, 1 (2016).
[36] EFT for 2D dilute bosons: LS, PRL **118**, 130402 (2017).

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