Quantum fluctuations
and vortex-antivortex unbinding
in the 2D BCS-BEC crossover

Luca Salasnich

Dipartimento di Fisica e Astronomia “Galileo Galilei” and CNISM, Università di Padova
INO-CNR, Research Unit of Sesto Fiorentino, Consiglio Nazionale delle Ricerche

Ischia, June 26, 2016

Work done in collaboration with
Giacomo Bighin and Flavio Toigo
Summary

- BCS-BEC crossover in 2D
- Quantum fluctuations in 2D
- New results for 2D BCS-BEC crossover
- Conclusions
In 2004 the 3D BCS-BEC crossover has been observed with ultracold gases made of two-component fermionic $^{40}\text{K}$ or $^6\text{Li}$ alkali-metal atoms.\(^1\)

This crossover is obtained by using a Fano-Feshbach resonance to change the 3D s-wave scattering length $a_s$ of the inter-atomic potential

$$a_s = a_{bg} \left( 1 + \frac{\Delta B}{B - B_0} \right), \quad (1)$$

where $B$ is the external magnetic field.

\(^1\)C.A. Regal et al., PRL 92, 040403 (2004); M.W. Zwierlein et al., PRL 92, 120403 (2004); J. Kinast et al., PRL 92, 150402 (2004).
Recently also the 2D BEC-BEC crossover has been achieved experimentally\(^2\) with a Fermi gas of two-component \(^6\)Li atoms. In 2D attractive fermions always form biatomic molecules with bound-state energy

\[
\epsilon_B \simeq \frac{\hbar^2}{ma_s^2},
\]

where \(a_s\) is the 2D s-wave scattering length, which is experimentally tuned by a Fano-Feshbach resonance.

The fermionic single-particle spectrum is given by

\[
E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta^2},
\]

where \(\Delta\) is the energy gap and \(\mu\) is the chemical potential: \(\mu > 0\) corresponds to the BCS regime while \(\mu < 0\) corresponds to the BEC regime. Moreover, in the deep BEC regime \(\mu \to -\epsilon_B/2\).

To study the 2D BCS-BEC crossover we adopt the formalism of functional integration. The partition function $Z$ of the uniform system with fermionic fields $\psi_s(r, \tau)$ at temperature $T$, in a $D$-dimensional volume $L^D$, and with chemical potential $\mu$ reads

$$Z = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{S}{\hbar} \right\},$$

where $(\beta \equiv 1/(k_B T)$ with $k_B$ Boltzmann’s constant)

$$S = \int_0^{\hbar \beta} d\tau \int_{L^D} d^D r \mathcal{L}$$

is the Euclidean action functional with Lagrangian density

$$\mathcal{L} = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\uparrow \psi_\downarrow$$

where $g$ is the attractive strength $(g < 0)$ of the s-wave coupling.

$^3$N. Nagaosa, Quantum Field Theory in Condensed Matter Physics (Springer, 1999)
Through the usual **Hubbard-Stratonovich transformation** the Lagrangian density $\mathcal{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the **auxiliary complex scalar field** $\Delta(r, \tau)$. In this way the effective Euclidean Lagrangian density reads

$$\mathcal{L}_e = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + \bar{\Delta} \psi_\uparrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow - \frac{|\Delta|^2}{g} . \tag{7}$$

We investigate the effect of fluctuations of the gap field $\Delta(r, t)$ around its mean-field value $\Delta_0$ which may be taken to be real. For this reason we set

$$\Delta(r, \tau) = \Delta_0 + \eta(r, \tau) , \tag{8}$$

where $\eta(r, \tau)$ is the complex field which describes pairing fluctuations.
In particular, we are interested in the grand potential $\Omega$, given by

$$
\Omega = -\frac{1}{\beta} \ln (Z) \simeq -\frac{1}{\beta} \ln (Z_{mf} Z_g) = \Omega_{mf} + \Omega_B ,
$$

where

$$
Z_{mf} = \int \mathcal{D} [\psi_s, \bar{\psi}_s] \exp \left\{ - \frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar} \right\}
$$

is the mean-field partition function and

$$
Z_g = \int \mathcal{D} [\psi_s, \bar{\psi}_s] \mathcal{D} [\eta, \bar{\eta}] \exp \left\{ - \frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar} \right\}
$$

is the partition function of Gaussian pairing fluctuations.
One finds that in the gas of paired fermions there are two kinds of elementary excitations: fermionic single-particle excitations with energy

$$E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2} ,$$  \hspace{1cm} (12)

where $\Delta_0$ is the pairing gap, and bosonic collective excitations with energy

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m} \left( \lambda \frac{\hbar^2 q^2}{2m} + 2 \frac{m c_s^2}{2} \right)} ,$$  \hspace{1cm} (13)

where $\lambda$ is the first correction to the familiar low-momentum phonon dispersion $E_{col}(q) \approx c_s \hbar q$ and $c_s$ is the sound velocity. Notice that both $\lambda$ and $c_s$ depend on the chemical potential $\mu$. 


Moreover, at the Gaussian level, the total grand potential reads

$$\Omega = \Omega_{mf} + \Omega_g ,$$  \hspace{1cm} (14)

where

$$\Omega_{mf} = \Omega_0 + \Omega_F^{(0)} + \Omega_F^{(T)}$$  \hspace{1cm} (15)

is the mean-field grand potential with

$$\Omega_0 = -\frac{\Delta_0^2}{g} L^D$$  \hspace{1cm} (16)

the grand potential of the order parameter $\Delta_0$,

$$\Omega_F^{(0)} = -\sum_k \left( E_{sp}(k) - \frac{\hbar^2 k^2}{2m} + \mu \right)$$  \hspace{1cm} (17)

the zero-point energy of fermionic single-particle excitations,

$$\Omega_F^{(T)} = \frac{2}{\beta} \sum_k \ln \left( 1 + e^{-\beta E_{sp}(k)} \right)$$  \hspace{1cm} (18)

the finite-temperature grand potential of the fermionic single-particle excitations.
The grand-potential of bosonic Gaussian fluctuations reads

\[
\Omega_g = \Omega_{g,B}^{(0)} + \Omega_{g,B}^{(T)} ,
\]

where

\[
\Omega_{g,B}^{(0)} = \frac{1}{2} \sum_q E_{col}(q)
\]

is the zero-point energy of bosonic collective excitations and

\[
\Omega_{g,B}^{(T)} = \frac{1}{\beta} \sum_q \ln \left(1 - e^{-\beta E_{col}(q)}\right)
\]

is the finite-temperature grand potential of the bosonic collective excitations.

Both \(\Omega_{F}^{(0)}\) and \(\Omega_{g,B}^{(0)}\) are ultraviolet divergent in any dimension \(D\) \((D = 1, 2, 3)\) and the regularization of these divergent terms is complicated by the fact that one also must take into account the BCS-BEC crossover.
In the analysis of the two-dimensional attractive Fermi gas one must remember that, contrary to the 3D case, 2D realistic interatomic attractive potentials have always a bound state. In particular, the binding energy $\epsilon_B > 0$ of two fermions can be written in terms of the positive 2D fermionic scattering length $a_s$ as

$$\epsilon_B = \frac{4}{e^2 \gamma} \frac{\hbar^2}{m a_s^2},$$

(22)

where $\gamma = 0.577...$ is the Euler-Mascheroni constant. Moreover, the attractive (negative) interaction strength $g$ of s-wave pairing is related to the binding energy $\epsilon_B > 0$ of a fermion pair in vacuum by the expression

$$-\frac{1}{g} = \frac{1}{2L^2} \sum_k \frac{1}{\hbar^2 k^2} \left( \frac{k^2}{2m} + \frac{1}{2} \epsilon_B \right).$$

(23)

---

In the 2D BCS-BEC crossover, at zero temperature \((T = 0)\) the mean-field grand potential \(\Omega_{mf}\) can be written as\(^6\) \((\epsilon_B > 0)\)

\[
\Omega_{mf} = -\frac{m L^2}{2 \pi \hbar^2} \left( \mu + \frac{1}{2} \epsilon_B \right)^2 .
\] (24)

Using

\[
n = -\frac{1}{L^2} \frac{\partial \Omega_{mf}}{\partial \mu}
\] (25)

one immediately finds the chemical potential \(\mu\) as a function of the number density \(n = N/L^2\), i.e.

\[
\mu = \frac{\pi \hbar^2}{m} n - \frac{1}{2} \epsilon_B .
\] (26)

In the BCS regime, where \(\epsilon_B \ll \epsilon_F\) with \(\epsilon_F = \pi \hbar^2 n / m\), one finds \(\mu \approx \epsilon_F > 0\) while in the BEC regime, where \(\epsilon_B \gg \epsilon_F\) one has \(\mu \approx -\epsilon_B / 2 < 0\). 

In the deep BEC regime of the 2D BCS-BEC crossover, where the chemical potential $\mu$ becomes negative, performing regularization of zero-point fluctuations we have recently found$^7$ that the zero-temperature grand potential (including bosonic excitations) is

$$\Omega = -\frac{mL^2}{64\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2 \ln \left( \frac{\epsilon_B}{2(\mu + \frac{1}{2}\epsilon_B)} \right).$$

(27)

This is exactly Popov's equation of state of 2D Bose gas with chemical potential $\mu_B = 2(\mu + \epsilon_B/2)$ and mass $m_B = 2m$. In this way we have identified the two-dimensional scattering length $a_B$ of composite bosons as

$$a_B = \frac{1}{2^{1/2}e^{1/4}} a_s. \quad (28)$$

The value $a_B/a_s = 1/(2^{1/2}e^{1/4}) \approx 0.551$ is in full agreement with $a_B/a_s = 0.55(4)$ obtained by Monte Carlo calculations$^8$.

At zero temperature we compare\(^9\) the first sound velocity

\[ c_s = \sqrt{\frac{n}{m} \frac{\partial \mu}{\partial n}} = \sqrt{-\frac{n}{m} \left( \frac{1}{L^2} \frac{\partial^2 \Omega(\mu)}{\partial \mu^2} \right)^{-1}}. \] (29)

with available experimental data\(^10\) (blue circles and red squares).

\(^9\)G. Bighin and LS, PRB 93, 014519 (2016).
New results for 2D BCS-BEC crossover (V)

The Berezinskii-Kosterlitz-Thouless critical temperature $T_{BKT}$ is determined by the jump of the renormalized superfluid density $n_{s,r}(T)$, derived$^{11}$ starting from the bare superfluid density

$$n_s(T) = n - \beta \int \frac{d^2k}{(2\pi)^2} k^2 \frac{e^{\beta E_{sp}(k)}}{(e^{\beta E_{sp}(k)} + 1)^2} - \frac{\beta}{2} \int \frac{d^2q}{(2\pi)^2} q^2 \frac{e^{\beta E_{col}(q)}}{(e^{\beta E_{col}(q)} - 1)^2}$$

(30)

and using Kosterlitz’s renormalization-group equations.$^{12}$

$^{11}$G. Bighin and LS, in preparation.

In fact the low-energy Hamiltonian of a fermionic superfluid can be recast\textsuperscript{13} as that of an effective continuous 2D XY model

\begin{equation}
H = \frac{J(T)}{2} \int d^2 r \left( \nabla \theta(r) \right)^2 ,
\end{equation}

where \( \theta(r) \) is the phase angle of the pairing field \( \Delta(r) = |\Delta(r)| e^{i\theta(r)} \) and

\begin{equation}
J(T) = \frac{\hbar^2}{4m} n_s(T)
\end{equation}

is the phase stiffness. The compactness of the phase angle \( \theta(r) \) implies that

\begin{equation}
\oint \nabla \theta(r) \cdot d\mathbf{r} = 2\pi q ,
\end{equation}

where \( q \) is the integer number associated to quantum vortices \((q > 0)\) and antivortices \((q < 0)\), which renormalize\textsuperscript{14} the phase stiffness and consequently also the superfluid density.

New results for 2D BCS-BEC crossover (VII)

Theoretical predictions for the Berezinskii-Kosterlitz-Thouless critical temperature $T_{BKT}$ (at which vortex-antivortex pairs unbind) compared to recent experimental observation (circles with error bars).

---

$^{15}$G. Bighin and LS, PRB 93, 014519 (2016); G. Bighin and LS, in preparation.

Conclusions

- The regularization of zero-point energy\textsuperscript{17} gives remarkable beyond-mean-field effects for composite bosons in the 2D BCS-BEC crossover at zero temperature:
  - logarithmic behavior of the equation of state
  - Bose-Bose scattering length $a_B$ vs Fermi-Fermi scattering length $a_s$
  - speed of first sound (and also second sound)
- Also at finite temperature beyond-mean-field effects, with the inclusion of quantized vortices and antivortices, become relevant in the strong-coupling regime of 2D BCS-BEC crossover:
  - superfluid density $n_s$
  - critical temperature $T_{BKT}$

\textsuperscript{17}For a very recent comprehensive review see:
Acknowledgements

Thank you for your attention!

Main sponsors: PRIN-MIUR, UNIPD.