Composite bosons in the 2D BCS-BEC crossover

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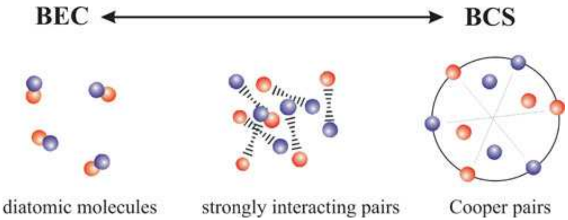
Summary

- BCS-BEC crossover with ultracold atoms
- Theory for a D-dimensional Fermi superfluid
- Results for the two-dimensional Fermi superfluid
- Conclusions

BCS-BEC crossover with ultracold atoms (I)

In 2004 the 3D BCS-BEC crossover has been observed with **ultracold gases made of fermionic ^{40}K and ^{6}Li alkali-metal atoms.**[1]



This crossover is obtained by changing (with a Feshbach resonance) the s-wave scattering length a_F of the inter-atomic potential:

– $a_F \to 0^-$ (BCS regime of weakly-interacting Cooper pairs)
– $a_F \to \pm\infty$ (unitarity limit of strongly-interacting Cooper pairs)
– $a_F \to 0^+$ (BEC regime of bosonic dimers)

[1] C.A. Regal *et al.*, PRL **92**, 040403 (2004); M.W. Zwierlein *et al.*, PRL **92**, 120403 (2004); M. Bartenstein *et al.*, PRL **92**, 120401 (2004); J. Kinast *et al.*, PRL **92**, 150402 (2004).

BCS-BEC crossover with ultracold atoms (II)

The crossover from a BCS superfluid ($a_F < 0$) to a BEC of molecular pairs ($a_F > 0$) has been investigated experimentally around a Feshbach resonance, where the s-wave scattering length a diverges, and it has been shown that the system is (meta)stable.
The detection of quantized vortices under rotation[2] has clarified that this dilute gas of ultracold atoms is superfluid.
Usually the BCS-BEC crossover is analyzed in terms of

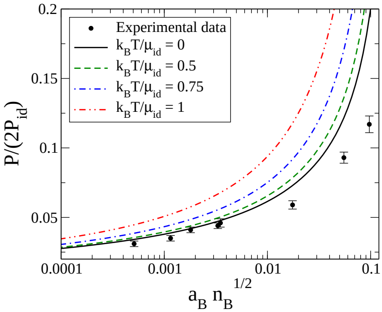
$$y = \frac{1}{k_F a_F} \quad (1)$$

the inverse scaled interaction strength, where $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wave number and n the total density.
The system is dilute because $r_e k_F \ll 1$, with r_e the effective range of the inter-atomic potential.

[2]M.W. Zwierlein *et al.*, Science **311**, 492 (2006); M.W. Zwierlein *et al.*, Nature **442**, 54 (2006)

BCS-BEC crossover with ultracold atoms (III)

In 2014 also the 2D BCS-BEC crossover has been achieved[3] with a **quasi-2D Fermi gas of ^{6}Li atoms** with widely tunable s-wave interaction, measuring the pressure P *vs* the gas parameter $a_B n_B^{1/2}$, with $a_B = a_F/(2^{1/2}e^{1/4})$ and $n_B = n/2$.



Filled circles with error bars are **experimental data** while lines are obtained with our beyond-mean-field finite-temperature theory[4].

[3] V. Makhalov, K. Martiyanov, and A. Turlapov, PRL **112**, 045301 (2014).
[4] LS and Toigo, PRA **91**, 011604(R) (2015); LS and F. Toigo, in preparation.

Theory for a D-dimensional Fermi superfluid (I)

We adopt the path integral formalism[5]. The partition function $\mathcal{Z}$ of the uniform system with fermionic fields $\psi_s(\mathbf{r},\tau)$ at temperature T, in a D-dimensional volume L^D, and with chemical potential μ reads

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \ \exp\left\{-\frac{1}{\hbar} \ S\right\} , \tag{2}$$

where ($\beta \equiv 1/(k_B T)$ with k_B Boltzmann's constant)

$$S = \int_0^{\hbar\beta} d\tau \int_{L^D} d^D\mathbf{r} \ \mathcal{L} \tag{3}$$

is the Euclidean action functional with Lagrangian density

$$\mathcal{L} = \bar{\psi}_s \left[\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu\right] \psi_s + \mathbf{g} \, \bar{\psi}_\uparrow \, \bar{\psi}_\downarrow \, \psi_\downarrow \, \psi_\uparrow \tag{4}$$

where $\mathbf{g}$ is the attractive strength ($\mathbf{g} < 0$) of the s-wave coupling.

[5] N. Nagaosa, Quantum Field Theory in Condensed Matter Physics (Springer, 1999).

Theory for a D-dimensional Fermi superfluid (II)

Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathcal{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(\mathbf{r}, \tau)$ so that:

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \mathcal{D}[\Delta, \bar{\Delta}] \; \exp\left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})}{\hbar} \right\} , \tag{5}$$

where

$$S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta}) = \int_0^{\hbar\beta} d\tau \int_{L^D} d^D\mathbf{r} \; \mathcal{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta}) \tag{6}$$

and the (exact) effective Euclidean Lagrangian density $\mathcal{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})$ reads

$$\mathcal{L}_e = \bar{\psi}_s \left[\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \psi_s + \bar{\Delta}\, \psi_\downarrow \, \psi_\uparrow + \Delta \bar{\psi}_\uparrow \, \bar{\psi}_\downarrow - \frac{|\Delta|^2}{\mathbf{g}} \, . \tag{7}$$

Theory for a D-dimensional Fermi superfluid (III)

We want to investigate the effect of fluctuations of the gap field $\Delta(\mathbf{r}, t)$ around its mean-field value Δ_0 which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r}, \tau) = \Delta_0 + \eta(\mathbf{r}, \tau) \ , \tag{8}$$

where $\eta(\mathbf{r}, \tau)$ is the complex field which describes pairing fluctuations. In particular, we are interested in the grand potential Ω, given by

$$\Omega = -\frac{1}{\beta} \ln\left(\mathcal{Z}\right) \simeq -\frac{1}{\beta} \ln\left(\mathcal{Z}_{mf} \mathcal{Z}_g\right) = \Omega_{mf} + \Omega_g \ , \tag{9}$$

where

$$\mathcal{Z}_{mf} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \ \exp\left\{-\frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar}\right\} \tag{10}$$

is the mean-field partition function and

$$\mathcal{Z}_g = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \mathcal{D}[\eta, \bar{\eta}] \ \exp\left\{-\frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar}\right\} \tag{11}$$

is the partition function of Gaussian pairing fluctuations.

Theory for a D-dimensional Fermi superfluid (IV)

To make a long story short, one finds that in the gas of paired fermions there are two kinds of elementary excitations: fermionic single-particle excitations with energy

$$E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2} \, , \qquad (12)$$

where Δ_0 is the pairing gap, and bosonic collective excitations with energy

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda \, \frac{\hbar^2 q^2}{2m} + 2 \; m \; c_s^2\right)} \, , \qquad (13)$$

where λ is the first correction to the familiar low-momentum phonon dispersion $E_{col}(q) \simeq c_s \hbar q$ and c_s is the sound velocity. Notice that both λ and c_s depend on the chemical potential μ.

Theory for a D-dimensional Fermi superfluid (V)

Moreover, at the Gaussian level, the total grand potential reads

$$\Omega = \Omega_{mf} + \Omega_{g} \ , \tag{14}$$

where

$$\Omega_{mf} = -\frac{\Delta_0^2}{\mathbf{g}} L^D + \Omega_F^{(0)} + \Omega_F^{(T)} \tag{15}$$

is the mean-field grand potential with

$$\Omega_F^{(0)} = -\sum_{\mathbf{k}} \left(E_{sp}(k) - \frac{\hbar^2 k^2}{2m} + \mu \right) \tag{16}$$

the zero-point energy of fermionic single-particle excitations,

$$\Omega_F^{(T)} = \frac{2}{\beta} \sum_{\mathbf{k}} \ln\left(1 + e^{-\beta E_{sp}(k)}\right) \tag{17}$$

the finite-temperature grand potential of the fermionic single-particle excitations.

Theory for a D-dimensional Fermi superfluid (VI)

The grand-potential of Gaussian fluctuations reads

$$\Omega_g = \Omega^{(0)}_{g,B} + \Omega^{(T)}_{g,B} \, , \tag{18}$$

where

$$\Omega^{(0)}_{g,B} = \frac{1}{2} \sum_{\mathbf{q}} E_{col}(q) \tag{19}$$

is the zero-point energy of bosonic collective excitations and

$$\Omega^{(T)}_{g,B} = \frac{1}{\beta} \sum_{\mathbf{q}} \ln\left(1 - e^{-\beta \, E_{col}(q)}\right) \tag{20}$$

is the finite-temperature grand potential of the bosonic collective excitations.

Both $\Omega^{(0)}_{F}$ and $\Omega^{(0)}_{g,B}$ are ultraviolet divergent in any dimension D ($D = 1, 2, 3$) and the regularization of these divergent terms is complicated by the fact that one also must take into account the BCS-BEC crossover.

Results of the two-dimensional Fermi superfluid (I)

In the analysis of the **two-dimensional attractive Fermi gas** one must remember that, contrary to the 3D case, 2D realistic interatomic attractive potentials have always a bound state. In particular[6], the binding energy $\epsilon_b > 0$ of two fermions can be written in terms of the positive 2D fermionic scattering length a_F as

$$\epsilon_b = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{m {a_F}^2} \;, \tag{21}$$

where $\gamma = 0.577...$ is the Euler-Mascheroni constant. Moreover, the attractive (negative) interaction strength $\mathbf{g}$ of s-wave pairing is related to the binding energy $\epsilon_b > 0$ of a fermion pair in vacuum by the expression[7]

$$-\frac{1}{\mathbf{g}} = \frac{1}{2L^2} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^2 k^2}{2m} + \frac{1}{2}\epsilon_b} \; . \tag{22}$$

[6] C. Mora and Y. Castin, PRA **67**, 053615 (2003.

[7] M. Randeria, J-M. Duan, and L-Y. Shieh, PRL **62**, 981 (1989).

Results of the two-dimensional Fermi superfluid (II)

In the **2D BCS-BEC crossover**, at zero temperature ($T = 0$) the mean-field grand potential Ω_{mf} can be written as[8] ($\epsilon_b > 0$)

$$\Omega_{mf} = -\frac{mL^2}{2\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_b)^2 \ . \tag{23}$$

Using

$$n = -\frac{1}{L^2}\frac{\partial\Omega_{mf}}{\partial\mu} \tag{24}$$

one immediately finds the chemical potential μ as a function of the number density $n = N/L^2$, i.e.

$$\mu = \frac{\pi\hbar^2}{m}n - \frac{1}{2}\epsilon_b \ . \tag{25}$$

In the BCS regime, where $\epsilon_b \ll \epsilon_F$ with $\epsilon_F = \pi\hbar^2 n/m$, one finds $\mu \simeq \epsilon_F > 0$ while in the BEC regime, where $\epsilon_b \gg \epsilon_F$ one has $\mu \simeq -\epsilon_b/2 < 0$.

[8] M. Randeria, J-M. Duan, and L-Y. Shieh, PRL **62**, 981 (1989).

Results of the two-dimensional Fermi superfluid (III)

Performing dimensional regularization of Gaussian fluctuations, we have recently found[9] that the zero-temperature total grand potential is

$$\Omega = \Omega_{mf} + \Omega_g = -\frac{mL^2}{\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_b)^2 \ \ln\left(\frac{\epsilon_b}{2(\mu + \frac{1}{2}\epsilon_b)}\right) . \quad (26)$$

in the deep BEC regime.
Introducing $\mu_B = 2(\mu + \epsilon_b/2)$ as the chemical potential of composite bosons with mass $m_B = 2m$ and density $n_B = n/2$, the zero-temperature total grand potential can be rewritten as

$$\Omega = -\frac{m_B L^2}{8\pi\hbar^2}\mu_B^2 \ \ln\left(\frac{\epsilon_0}{\mu_B}\right) , \quad (27)$$

that is exactly the Popov equation of state of 2D weakly-interacting bosons[10]

[9] LS and F. Toigo, PRA **91**, 011604(R) (2015).
[10] V.N. Popov, Theor. Math. Phys. A **11**, 565 (1972).

Results of the two-dimensional Fermi superfluid (IV)

provided that we identify the parameter

$$\epsilon_0 = \frac{4}{e^{2\gamma+1/2}} \frac{\hbar^2}{m_B {a_B}^2} \tag{28}$$

of the Popov theory of bosons (with scattering length a_B)[11] with the binding energy

$$\epsilon_b = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{m {a_F}^2} \tag{29}$$

of paired fermions (with scattering length a_F).[12] Thus, we find[13]

$$\boxed{a_B = \frac{1}{2^{1/2} e^{1/4}} \; a_F \; .} \tag{30}$$

The value $a_B/a_F = 1/(2^{1/2}e^{1/4}) \simeq 0.551$ is in full agreement with that ($a_B/a_F = 0.55(4)$) obtained by Monte Carlo calculations[14].

[11] C. Mora and Y. Castin, PRL **102**, 180404 (2009).
[12] C. Mora and Y. Castin, PRA **67**, 053615 (2003).
[13] LS and F. Toigo, PRA **91**, 011604(R) (2015).
[14] G. Bertaina and S. Giorgini, PRL **106**, 110403 (2011).

Results of the two-dimensional Fermi superfluid (V)

At finite temperature ($T \neq 0$) the pressure P is immediately obtained using the thermodynamic formula $P = -\Omega/L^2$. Taking into account that the main thermal contribution is due to collective bosonic excitatons we get[15]

$$P = \frac{m_B}{8\pi\hbar^2}\,\mu_B^2\left[\ln\left(\frac{\epsilon_0}{\mu_B}\right) + 4\zeta(3)\left(\frac{k_B T}{\mu_B}\right)^3\right] , \qquad (31)$$

and also, by using $n_B = \left(\frac{\partial\Omega}{\partial\mu_B}\right)_{T,L^2}$,

$$n_B = \frac{m_B}{4\pi\hbar^2}\,\mu_B\left[\ln\left(\frac{\epsilon_0}{\mu_B\,e^{1/2}}\right) - 2\zeta(3)\left(\frac{k_B T}{\mu_B}\right)^3\right] \qquad (32)$$

where $\zeta(x)$ is the Riemman zeta fuction and $\zeta(3) = 1.20205$. Eqs. (31) and (32) give, at fixed $k_B T/\mu_B$, a parametric formula for the the pressure P as a function of the density n_B where μ_B is the dummy parameter.

[15] LS and F. Toigo, in preparation

Conclusions

- The D-dimensional superfluid Fermi gas in the BCS-BEC crossover has a divergent zero-point energy due to:
  – fermionic single-particle excitations (mean-field)
  – bosonic collective excitations (Gaussian fluctuations).
- **Regularization** of the divergent zero-point energy gives remarkable analytical results for composite bosons in two dimensions[16]:
  – reliable 2D equation of state (Popov);
  – analytical formula connecting a_B and a_F.
- Notice that also in three-dimensions one can regularize the divergent zero-point energy due to fermionic and bosonic excitations[17]

[16] LS and Toigo, PRA **91**, 011604(R) (2015).

[17] P. Pieri and G. Strinati, PRB **61**, 15370 (2000); H.Hu, X.-J. Liu, and P. Drummond, EPL **74**, 574 (2006); R.B. Diener, R. Sensarma, and M. Randeria, PRA **77**, 023626 (2008); LS and Bighin, PRA **91**, 033610 (2015).