Composite bosons in the 2D BCS-BEC crossover

Luca Salasnich

Dipartimento di Fisica e Astronomia “Galileo Galilei” and CNISM, Università di Padova

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Collaboration with Flavio Toigo (Univ. of Padova)
Summary

- BCS-BEC crossover with ultracold atoms
- Theory for a $D$-dimensional Fermi superfluid
- Results for the two-dimensional Fermi superfluid
- Conclusions
In 2004 the 3D BCS-BEC crossover has been observed with ultracold gases made of fermionic $^{40}\text{K}$ and $^6\text{Li}$ alkali-metal atoms.\(^1\)

This crossover is obtained by changing (with a Feshbach resonance) the s-wave scattering length \(a_F\) of the inter-atomic potential:
- \(a_F \to 0^-\) (BCS regime of weakly-interacting Cooper pairs)
- \(a_F \to \pm \infty\) (unitarity limit of strongly-interacting Cooper pairs)
- \(a_F \to 0^+\) (BEC regime of bosonic dimers)

\(^1\)C.A. Regal et al., PRL 92, 040403 (2004); M.W. Zwierlein et al., PRL 92, 120403 (2004); M. Bartenstein et al., PRL 92, 120401 (2004); J. Kinast et al., PRL 92, 150402 (2004).
BCS-BEC crossover with ultracold atoms (II)

The crossover from a BCS superfluid ($a_F < 0$) to a BEC of molecular pairs ($a_F > 0$) has been investigated experimentally around a Feshbach resonance, where the s-wave scattering length $a$ diverges, and it has been shown that the system is (meta)stable. The detection of quantized vortices under rotation$^2$ has clarified that this dilute gas of ultracold atoms is superfluid. Usually the BCS-BEC crossover is analyzed in terms of

$$y = \frac{1}{k_F a_F}$$ (1)

the inverse scaled interaction strength, where $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wave number and $n$ the total density. The system is dilute because $r_e k_F \ll 1$, with $r_e$ the effective range of the inter-atomic potential.

In 2014 also the 2D BCS-BEC crossover has been achieved\(^3\) with a quasi-2D Fermi gas of \(^6\)Li atoms with widely tunable s-wave interaction, measuring the pressure \(P\) vs the gas parameter \(a_B n_B^{1/2}\), with \(a_B = a_F / (2^{1/2} e^{1/4})\) and \(n_B = n/2\).

Filled circles with error bars are experimental data while lines are obtained with our beyond-mean-field finite-temperature theory\(^4\).

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\(^3\)V. Makhalov, K. Martiyanov, and A. Turlapov, PRL 112, 045301 (2014).

\(^4\)LS and Toigo, PRA 91, 011604(R) (2015); LS and F. Toigo, in preparation.
We adopt the path integral formalism\(^5\). The partition function \(Z\) of the uniform system with fermionic fields \(\psi_s(r, \tau)\) at temperature \(T\), in a \(D\)-dimensional volume \(L^D\), and with chemical potential \(\mu\) reads

\[
Z = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{1}{\hbar} S \right\}, \tag{2}
\]

where \((\beta \equiv 1/(k_B T)\) with \(k_B\) Boltzmann’s constant) \(S = \int_0^{\hbar \beta} d\tau \int_{L^D} d^D r \mathcal{L}\) is the Euclidean action functional with Lagrangian density

\[
\mathcal{L} = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + g \bar{\psi}^\uparrow \bar{\psi}^\downarrow \psi^\downarrow \psi^\uparrow \tag{4}
\]

where \(g\) is the attractive strength \((g < 0)\) of the s-wave coupling.

\(^5\)N. Nagaosa, Quantum Field Theory in Condensed Matter Physics (Springer, 1999).
Theory for a $D$-dimensional Fermi superfluid (II)

Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathcal{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(r, \tau)$ so that:

$$
Z = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \mathcal{D}[\Delta, \bar{\Delta}] \exp \left\{ - \frac{S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})}{\hbar} \right\},
$$

(5)

where

$$
S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta}) = \int_0^{\hbar \beta} d\tau \int L^D \mathcal{D}[\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})
$$

(6)

and the (exact) effective Euclidean Lagrangian density $\mathcal{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})$ reads

$$
\mathcal{L}_e = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + \bar{\Delta} \psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow - \frac{|\Delta|^2}{g}.
$$

(7)
We want to investigate the effect of fluctuations of the gap field $\Delta(\mathbf{r}, t)$ around its mean-field value $\Delta_0$ which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r}, \tau) = \Delta_0 + \eta(\mathbf{r}, \tau),$$

where $\eta(\mathbf{r}, \tau)$ is the complex field which describes pairing fluctuations. In particular, we are interested in the grand potential $\Omega$, given by

$$\Omega = -\frac{1}{\beta} \ln (\mathcal{Z}) \simeq -\frac{1}{\beta} \ln (\mathcal{Z}_{mf} \mathcal{Z}_g) = \Omega_{mf} + \Omega_g,$$

where

$$\mathcal{Z}_{mf} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar} \right\}$$

is the mean-field partition function and

$$\mathcal{Z}_g = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \mathcal{D}[\eta, \bar{\eta}] \exp \left\{ -\frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar} \right\}$$

is the partition function of Gaussian pairing fluctuations.
To make a long story short, one finds that in the gas of paired fermions there are two kinds of elementary excitations: fermionic single-particle excitations with energy

\[ E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2}, \]  

(12)

where \( \Delta_0 \) is the pairing gap, and bosonic collective excitations with energy

\[ E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m} \left(\lambda \frac{\hbar^2 q^2}{2m} + 2m c_s^2\right)}, \]  

(13)

where \( \lambda \) is the first correction to the familiar low-momentum phonon dispersion \( E_{col}(q) \approx c_s \hbar q \) and \( c_s \) is the sound velocity. Notice that both \( \lambda \) and \( c_s \) depend on the chemical potential \( \mu \).
Moreover, at the Gaussian level, the total grand potential reads

$$\Omega = \Omega_{mf} + \Omega_g ,$$

(14)

where

$$\Omega_{mf} = - \frac{\Delta_0^2}{g} L^D + \Omega_F^{(0)} + \Omega_F^{(T)}$$

(15)

is the mean-field grand potential with

$$\Omega_F^{(0)} = - \sum_k \left( E_{sp}(k) - \frac{\hbar^2 k^2}{2m} + \mu \right)$$

(16)

the zero-point energy of fermionic single-particle excitations,

$$\Omega_F^{(T)} = \frac{2}{\beta} \sum_k \ln \left( 1 + e^{-\beta E_{sp}(k)} \right)$$

(17)

the finite-temperature grand potential of the fermionic single-particle excitations.
The grand-potential of Gaussian fluctuations reads

$$\Omega_g = \Omega_g^{(0)} + \Omega_g^{(T)} ,$$  \hspace{1cm} (18)

where

$$\Omega_g^{(0)} = \frac{1}{2} \sum_q E_{\text{col}}(q)$$  \hspace{1cm} (19)

is the zero-point energy of bosonic collective excitations and

$$\Omega_g^{(T)} = \frac{1}{\beta} \sum_q \ln (1 - e^{-\beta E_{\text{col}}(q)})$$  \hspace{1cm} (20)

is the finite-temperature grand potential of the bosonic collective excitations.

Both $\Omega_F^{(0)}$ and $\Omega_g^{(0)}$ are ultraviolet divergent in any dimension $D$ ($D = 1, 2, 3$) and the regularization of these divergent terms is complicated by the fact that one also must take into account the BCS-BEC crossover.
In the analysis of the **two-dimensional attractive Fermi gas** one must remember that, contrary to the 3D case, 2D realistic interatomic attractive potentials have always a bound state. In particular\(^6\), the binding energy \(\epsilon_b > 0\) of two fermions can be written in terms of the positive 2D fermionic scattering length \(a_F\) as

\[
\epsilon_b = \frac{4}{e^2\gamma} \frac{\hbar^2}{ma_F^2},
\]

where \(\gamma = 0.577...\) is the Euler-Mascheroni constant. Moreover, the attractive (negative) interaction strength \(g\) of s-wave pairing is related to the binding energy \(\epsilon_b > 0\) of a fermion pair in vacuum by the expression\(^7\)

\[
-\frac{1}{g} = \frac{1}{2L^2} \sum_k \frac{1}{\hbar^2k^2/2m + \frac{1}{2}\epsilon_b}.
\]

\(^7\)M. Randeria, J-M. Duan, and L-Y. Shieh, PRL 62, 981 (1989).
In the 2D BCS-BEC crossover, at zero temperature (\( T = 0 \)) the mean-field grand potential \( \Omega_{mf} \) can be written as\(^8\) (\( \epsilon_b > 0 \))

\[
\Omega_{mf} = -\frac{m L^2}{2 \pi \hbar^2} \left( \mu + \frac{1}{2} \epsilon_b \right)^2 .
\] (23)

Using

\[
n = -\frac{1}{L^2} \frac{\partial \Omega_{mf}}{\partial \mu}
\] (24)

one immediately finds the chemical potential \( \mu \) as a function of the number density \( n = N/L^2 \), i.e.

\[
\mu = \frac{\pi \hbar^2}{m} n - \frac{1}{2} \epsilon_b .
\] (25)

In the BCS regime, where \( \epsilon_b \ll \epsilon_F \) with \( \epsilon_F = \pi \hbar^2 n/m \), one finds \( \mu \simeq \epsilon_F > 0 \) while in the BEC regime, where \( \epsilon_b \gg \epsilon_F \) one has \( \mu \simeq -\epsilon_b/2 < 0 \).

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\(^8\)M. Randeria, J-M. Duan, and L-Y. Shieh, PRL 62, 981 (1989).
Performing dimensional regularization of Gaussian fluctuations, we have recently found\(^9\) that the zero-temperature total grand potential is

\[
\Omega = \Omega_{mf} + \Omega_g = \frac{mL^2}{\pi\hbar^2} \left(\mu + \frac{1}{2}\epsilon_b\right)^2 \ln \left(\frac{\epsilon_b}{2(\mu + \frac{1}{2}\epsilon_b)}\right) . \tag{26}
\]

in the deep BEC regime.

Introducing \(\mu_B = 2(\mu + \epsilon_b/2)\) as the chemical potential of composite bosons with mass \(m_B = 2m\) and density \(n_B = n/2\), the zero-temperature total grand potential can be rewritten as

\[
\Omega = -\frac{m_B L^2}{8\pi\hbar^2} \mu_B^2 \ln \left(\frac{\epsilon_0}{\mu_B}\right) , \tag{27}
\]

that is exactly the Popov equation of state of 2D weakly-interacting bosons\(^{10}\)

provided that we identify the parameter

\[ \epsilon_0 = \frac{4}{e^{2\gamma+1/2}} \frac{\hbar^2}{m_B a_B^2} \]  

(28)

of the Popov theory of bosons (with scattering length \(a_B\))\(^{11}\) with the binding energy

\[ \epsilon_b = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{ma_F^2} \]  

(29)

of paired fermions (with scattering length \(a_F\)).\(^{12}\) Thus, we find\(^{13}\)

\[ a_B = \frac{1}{2^{1/2}e^{1/4}} a_F . \]  

(30)

The value \(a_B/a_F = 1/(2^{1/2}e^{1/4}) \sim 0.551\) is in full agreement with that (\(a_B/a_F = 0.55(4)\)) obtained by Monte Carlo calculations\(^{14}\).

\(^{11}\)C. Mora and Y. Castin, PRL 102, 180404 (2009).
\(^{13}\)LS and F. Toigo, PRA 91, 011604(R) (2015).
At finite temperature \((T \neq 0)\) the pressure \(P\) is immediately obtained using the thermodynamic formula \(P = -\Omega/L^2\). Taking into account that the main thermal contribution is due to collective bosonic excitations we get\(^{15}\)

\[
P = \frac{m_B}{8\pi \hbar^2} \mu_B^2 \left[ \ln \left( \frac{\epsilon_0}{\mu_B} \right) + 4\zeta(3) \left( \frac{k_B T}{\mu_B} \right)^3 \right],
\]

(31)

and also, by using \(n_B = \left( \frac{\partial \Omega}{\partial \mu_B} \right)_{T,L^2}\),

\[
n_B = \frac{m_B}{4\pi \hbar^2} \mu_B \left[ \ln \left( \frac{\epsilon_0}{\mu_B e^{1/2}} \right) - 2\zeta(3) \left( \frac{k_B T}{\mu_B} \right)^3 \right]
\]

(32)

where \(\zeta(x)\) is the Riemann zeta function and \(\zeta(3) = 1.20205\). Eqs. (31) and (32) give, at fixed \(k_B T/\mu_B\), a parametric formula for the the pressure \(P\) as a function of the density \(n_B\) where \(\mu_B\) is the dummy parameter.

\(^{15}\)LS and F. Toigo, in preparation
Conclusions

- The $D$-dimensional superfluid Fermi gas in the BCS-BEC crossover has a divergent zero-point energy due to:
  - fermionic single-particle excitations (mean-field)
  - bosonic collective excitations (Gaussian fluctuations).
- **Regularization** of the divergent zero-point energy gives remarkable analytical results for composite bosons in two dimensions$^{16}$:
  - reliable 2D equation of state (Popov);
  - analytical formula connecting $a_B$ and $a_F$.
- Notice that also in three-dimensions one can regularize the divergent zero-point energy due to fermionic and bosonic excitations$^{17}$

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