Superfluid density, sound velocity and Goldstone mode in the 2D BCS-BEC crossover

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Summary

- Condensation and superfluidity in 2D systems
- 2D Fermi gas with pairing
- Mean-field
- Zero-temperature
- Finite-temperature
- Beyond mean-field
- Open problems
According to the **Mermin-Wagner theorem**\(^1\) in a 2D uniform system one can find **true condensation**, i.e. off-diagonal-long-range-order (ODLRO), only at zero temperature \((T = 0)\).

Nevertheless, as shown by Hohenberg\(^2\) the 2D uniform system can have **quasi condensation**, i.e. algebraic-long-range-order (ALRO), below a critical finite temperature. This critical temperature is usually identified with the Berezinskii-Kosterlitz-Thouless temperature\(^3\) below which the 2D system has a finite **superfluidity**.

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We consider a **2D neutral Fermi gas with attractive s-wave interaction**. The **partition function** $Z$ of the system at temperature $T$, in a region of area $L^2$, and with chemical potential $\mu$ can be written as

$$Z = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{1}{\hbar} S \right\}, \quad (1)$$

where

$$S = \int_0^{\hbar\beta} d\tau \int_{L^2} d^2r \mathcal{L} \quad (2)$$

is the **Euclidean action functional** and $\mathcal{L}$ is given by

$$\mathcal{L} = (\bar{\psi}_\uparrow, \bar{\psi}_\downarrow) \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \left( \begin{array}{c} \psi_\uparrow \\ \psi_\downarrow \end{array} \right) + g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \quad (3)$$

with $g < 0$ is the attractive strength of the s-wave coupling. Notice that $\beta = 1/(k_B T)$ with $k_B$ the Boltzmann constant.
The Lagrangian density $\mathcal{L}$ is quartic in the fermionic fields $\psi_s$, but one can reduce the problem to a quadratic Lagrangian density by introducing an auxiliary complex scalar field $\Delta(r, \tau)$ via Hubbard-Stratonovich transformation\(^4\), which gives

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \mathcal{D}[\Delta, \bar{\Delta}] \exp \left\{ -\frac{S_e}{\hbar} \right\}, \quad (4)$$

where

$$S_e = \int_0^{\hbar \beta} d\tau \int L^2 d^2r \mathcal{L}_e \quad (5)$$

and the (exact) effective Euclidean Lagrangian density $\mathcal{L}_e$ reads

$$\mathcal{L}_e = (\bar{\psi}_\uparrow, \bar{\psi}_\downarrow) \left[ \hbar \partial_\tau - \hbar^2 \frac{\nabla^2}{2m} - \mu \right] \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} + \bar{\Delta} \psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow - \frac{|\Delta|^2}{g}. \quad (6)$$

It is a standard procedure to integrate out the quadratic fermionic fields and to get a new effective action $S_{\text{eff}}$ which depends only on the auxiliary field $\Delta(r, \tau)$. In this way we obtain

$$Z = \int \mathcal{D}[\Delta, \bar{\Delta}] \exp \left\{ -S_{\text{eff}} / \hbar \right\},$$

where

$$S_{\text{eff}} = - \text{Tr} [\ln (G^{-1})] - \int_0^{\hbar \beta} d\tau \int_{L^2} d^2 r \frac{|\Delta|^2}{g}$$

with

$$G^{-1} = \left( \begin{array}{cc} \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu & \Delta \\ \Delta & \hbar \partial_\tau + \frac{\hbar^2}{2m} \nabla^2 + \mu \end{array} \right)$$

We stress that at this level the effective action $S_{\text{eff}}$ is formally exact.
In the mean-field approximation one considers a constant and real gap parameter, i.e.
\[ \Delta(r, \tau) = \Delta_0, \]  
(10)

and the partition function becomes
\[ Z_{mf} = \exp \left\{ -\frac{S_{mf}}{\hbar} \right\} = \exp \left\{ -\beta \Omega_{mf} \right\}, \]  
(11)

where
\[ \Omega_{mf} = -\sum_k \frac{1}{\beta} \left[ 2 \ln(2 \cosh(\beta E_k/2)) - \beta \xi_k \right] - L^2 \frac{\Delta_0^2}{g} \]  
(12)

with \( \xi_k = \hbar^2 k^2/(2m) - \mu \) and
\[ E_k = \sqrt{\xi_k^2 + \Delta_0^2}. \]  
(13)
The constant and real gap parameter $\Delta_0$ is obtained from

$$\frac{\partial \Omega_{mf}}{\partial \Delta_0} = 0,$$

which gives the gap equation

$$-\frac{1}{g} = \frac{1}{L^2} \sum_k \frac{\tanh(\beta E_k/2)}{2E_k}.$$  \hspace{1cm} (15)

The integral on the right side of this equation is divergent. However, in two dimensions quite generally a bound-state energy $\epsilon_B$ exists. For the contact potential the bound-state equation is

$$-\frac{1}{g} = \frac{1}{\Omega} \sum_k \frac{1}{2\hbar^2 k^2 + \epsilon_B}.$$  \hspace{1cm} (16)
In this way one obtains the regularized gap equation\textsuperscript{5}

\[ \sum_{k} \left( \frac{\tanh (\beta E_k/2)}{\hbar^2 k^2/2m + \epsilon_B/2} - \frac{1}{E_k} \right) = 0, \quad (17) \]

which can be used to study the BCS-BEC crossover by varying the binding energy $\epsilon_B$.

We observe that the binding energy $\epsilon_B$ can be written as $\epsilon_B \simeq \hbar^2/(ma_{2D})$, where $a_{2D}$ is the 2D s-wave scattering length, such that $a_{2D} \simeq a_z \exp(-a_z/a_{3D})$ with $a_{3D}$ the 3D scattering length and $a_z$ the characteristic length of the transverse confinement.\textsuperscript{6}

From the thermodynamic formula

\[ N = - \left( \frac{\partial \Omega_{mf}}{\partial \mu} \right)_{L^2,T} \]  

(18)

we obtain the equation for the total number of fermions

\[ N = \sum_k \left( 1 - \frac{\xi_k}{E_k} \tanh \left( \frac{\beta E_k}{2} \right) \right) . \]  

(19)

Moreover, the equation for the \( T = 0 \) number of quasi-condensed fermionic atoms\(^7\) reads

\[ N_0 = 2 \int d^2r \ d^2r' \ |\langle \psi_{\downarrow}(r) \ \psi_{\uparrow}(r') \rangle|^2 = \sum_k \frac{\Delta_0^2}{2E_k^2} \tanh \left( \frac{\beta E_k}{2} \right) . \]  

(20)

Zero-temperature properties (I)

At $T = 0$ the grand potential is given by

$$\Omega_{mf} = -\frac{m}{4\pi\hbar^2}L^2 \left( \mu^2 + \mu \sqrt{\mu^2 + \Delta_0^2} \right), \quad (21)$$

where the chemical potential $\mu$ reads

$$\mu = \epsilon_F - \frac{1}{2}\epsilon_B, \quad (22)$$

with $\epsilon_F = \pi\hbar^2 n/m$ the 2D Fermi energy, and the gap parameter $\Delta_0$ is instead

$$\Delta_0 = \sqrt{2\epsilon_F \epsilon_B}. \quad (23)$$

In addition, we find$^8$ this nice formula for the condensate fraction

$$\frac{N_0}{N} = \frac{1}{2} \left\{ \frac{\pi}{2} + \arctan \left( \frac{\mu}{\Delta} \right) \right\} \left( \frac{\mu}{\Delta} + \sqrt{1 + \left( \frac{\mu}{\Delta} \right)^2} \right). \quad (24)$$

Zero-temperature properties (II)

Figure: Upper panel: chemical potential $\mu$ and energy gap $\Delta_0$ as a function of the binding energy $\epsilon_B$ of pairs. Lower panel: Bose-condensate fraction $N_0/N$ of fermionic atoms as a function of the binding energy $\epsilon_B$ of pairs.
Zero-temperature properties (III)

According to Landau\textsuperscript{9} the first sound velocity $c_s$ is given by

$$m c_s^2 = \left( \frac{\partial P}{\partial n} \right)_{L^2, \bar{S}},$$

where $P$ is the pressure and $\bar{S} = S/N$ is the entropy per particle of the superfluid. Moreover, at zero temperature it holds the following equality

$$\left( \frac{\partial P}{\partial n} \right)_{L^2, 0} = n \left( \frac{\partial \mu}{\partial n} \right)_{L^2}.$$

Using the 2D zero-temperature mean-field result

$$\mu = \epsilon_F - \frac{1}{2} \epsilon_B,$$

where $\epsilon_F = (\pi \hbar^2 / m)n = mv_F^2 / 2$, we finally obtain

$$c_s = \frac{v_F}{\sqrt{2}}.$$

\textsuperscript{9}L.D. Landau, Journal of Physics USSR 5, 71 (1941).
One can explicitly calculate the temperature $T^*$ at which $\Delta_0 = 0$. In particular, one obtains\textsuperscript{10} the following equations

\begin{equation}
\mu(T^*) = k_B T^* \ln \left( e^{\epsilon_F/(k_B T^*)} - 1 \right),
\end{equation}

\begin{equation}
\epsilon_B = k_B T^* \frac{\pi}{\gamma} \exp \left( - \int_0^{\mu(T^*)/(2k_B T^*)} \frac{\tanh(u)}{u} du \right),
\end{equation}

which determine $T^*$ and $\mu(T^*)$ as a function of the binding energy $\epsilon_B$, with $\gamma = 1.781$.

Figure: Critical temperature $T^*$ (solid line), critical chemical potential $\mu(T^*)$ (dashed line), and zero-temperature chemical potential $\mu(0)$ as a function of the binding energy $\epsilon_B$ of pairs.
Let us now consider beyond mean-field effects. We have seen that the exact partition function can be written as

$$Z = \int \mathcal{D}[\Delta, \bar{\Delta}] \exp \left\{ -\frac{S_{\text{eff}}[\Delta, \bar{\Delta}]}{\hbar} \right\},$$

(31)

where $S_{\text{eff}}[\Delta, \bar{\Delta}]$ is the effective action, which is a functional of the complex bosonic auxiliary field $\Delta(r, \tau)$ of pairing. We impose that

$$\Delta(r, \tau) = (\Delta_0 + \sigma(r, \tau)) e^{i\theta(r, \tau)}.$$

(32)

The partition function can be then formally written as

$$Z = e^{-\beta \Omega_{mf}(\Delta_0)} \int \mathcal{D}[\sigma, \theta] \exp \left\{ -\frac{S_{bmf}[\sigma, \theta; \Delta_0]}{\hbar} \right\}.$$

(33)
Beyond mean-field (II)

Expanding $S_{bmf}[\sigma, \theta; \Delta_0]$ at the second order and functional-integrating over the amplitude field $\sigma(r, \tau)$ one obtains\textsuperscript{11}

$$Z = e^{-\beta \Omega_{mf}(\Delta_0)} \int \mathcal{D}[\theta] \exp \left\{ -S[\theta; \Delta_0]/\hbar \right\}, \quad (34)$$

where

$$S[\theta; \Delta_0] = \int_0^{\hbar \beta} d\tau \int 2 \, d^2r \left\{ \frac{J}{2} (\nabla \theta)^2 + \frac{K}{2} (\partial_\tau \theta)^2 \right\} \quad (35)$$

is the action functional of the phase field (Goldstone field) with $J$ the phase stiffness and $K$ the phase susceptibility.

At $T = 0$ we find

$$J = \frac{\epsilon_F}{4\pi}, \quad K = \frac{m}{4\pi}, \quad (36)$$

and the velocity $c_\theta$ of the Goldstone field reads

$$c_\theta = \sqrt{\frac{J}{K}} = \frac{v_F}{\sqrt{2}} = c_s. \quad (37)$$

Beyond mean-field (III)

Figure: Upper panel: 2D scaled sound velocity $c_s/v_F$ vs scaled binding energy $\epsilon_B/\epsilon_F$. Lower panel: 3D scaled sound velocity $c_s/v_F$ vs scaled inverse interaction strength $1/(k_F a)$.
The renormalization-group theory\textsuperscript{12} dictates that for our 2D system the superfluid density $n_s$ is zero above the Berezinskii-Kosterlitz-Thouless critical temperature $T_{BKT}$. Moreover below $T_{BKT}$ the superfluid density can be written as

\[ n_s(T) = \frac{4m}{\hbar^2} J(T) \quad \text{for} \ T < T_{BKT}, \quad (38) \]

and the critical temperature $T_{BKT}$ can be estimated by solving self-consistently

\[ k_B T_{BKT} = \frac{\pi}{2} J(T_{BKT}), \quad (39) \]

where $J(T)$ is the finite-temperature stiffness of our action functional $S_\theta$ of the phase.

Beyond mean-field (V)

**Figure:** Dashed line: temperature $T^*$ above which $\Delta_0$ is zero; solid line: Berezinskii-Kosterlitz-Thouless critical temperature $T_{BKT}$. 
Figure: Superfluid fraction $n_s/n$ as a function of the scaled temperature $T/T_{BKT}$ for different values of the scaled binding energy $\epsilon_B/\epsilon_F$, where $\epsilon_F = (\hbar^2/m)\pi n$ is the Fermi energy. Above $T_{BKT}$ one has $n_s = 0$. 
There are several open problems regarding our 2D Fermi superfluid in the BCS-BEC crossover. Among them we mention:

- first and second sound at finite temperature
- quasi-condensate at finite temperature
- beyond mean-field equation of state
- unbalanced system
THANK YOU FOR YOUR ATTENTION!

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