Condensate fraction for a polarized three-dimensional Fermi gas

Luca Salasnich

Dipartimento di Fisica e Astronomia “Galileo Galilei”, Università di Padova, Italy

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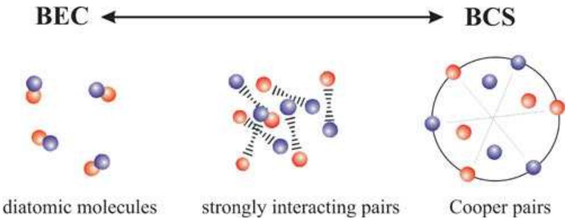
Collaboration with:
Giacomo Bighin, Luca Dell’Anna, Giovanni Mazzarella (Padova Univ.)
Nicola Manini (Milano Univ.), and Alberto Parola (Insubria Univ.)

Summary

- BCS-BEC crossover with ultracold atoms
- Uniform Fermi superfluid
- Trapped Fermi superfluid
- Polarized Fermi superfluid
- Open problems

BCS-BEC crossover with ultracold atoms (I)

In 2004 the BCS-BEC crossover has been observed with **ultracold gases made of two-component fermionic alkali-metal atoms.**[1]



This crossover is obtained by changing (with a Feshbach resonance) the s-wave scattering length a_s of the inter-atomic potential:

– $a_s \to 0^-$ (BCS regime of weakly-interacting Cooper pairs)
– $a_s \to \pm\infty$ (**unitarity limit** of strongly-interacting Cooper pairs)
– $a_s \to 0^+$ (BEC regime of bosonic dimers)

[1] C.A. Regal et al., PRL **92**, 040403 (2004); M.W. Zwierlein et al., PRL **92**, 120403 (2004); M. Bartenstein, A. Altmeyer et al., PRL **92**, 120401 (2004); J. Kinast et al., PRL **92**, 150402 (2004).

BCS-BEC crossover with ultracold atoms (II)

The crossover from a BCS superfluid ($a_s < 0$) to a BEC of molecular pairs ($a_s > 0$) has been investigated experimentally around a Feshbach resonance, where the s-wave scattering length a_s diverges, and it has been shown that the system is (meta)stable.
The detection of **quantized vortices** under rotation[2] has clarified that **this dilute and ultracold gas of Fermi atoms is superfluid**.
Usually the BCS-BEC crossover is analyzed in terms of

$$y = \frac{1}{k_F a_s} \qquad (1)$$

the inverse scaled interaction strength, where $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wave number and n the total density.
The system is dilute because $r_e k_F \ll 1$, with r_e the effective range of the inter-atomic potential.

[2] M.W. Zwierlein *et al.*, Science **311**, 492 (2006); M.W. Zwierlein *et al.*, Nature **442**, 54 (2006).

Uniform Fermi superfluid (I)

The shifted Hamiltonian of the uniform two-spin-component Fermi superfluid made of ultracold atoms is given by

$$\hat{H}' = \int d^3\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^+_\sigma(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu \right) \hat{\psi}_\sigma(\mathbf{r}) + g\, \hat{\psi}^+_\uparrow(\mathbf{r})\, \hat{\psi}^+_\downarrow(\mathbf{r})\, \hat{\psi}_\downarrow(\mathbf{r})\, \hat{\psi}_\uparrow(\mathbf{r})\,, \quad (2)$$

where $\hat{\psi}_\sigma(\mathbf{r})$ is the field operator that annihilates a fermion of spin σ in the position $\mathbf{r}$, while $\hat{\psi}^+_\sigma(\mathbf{r})$ creates a fermion of spin σ in $\mathbf{r}$. Here $g < 0$ is the strength of the attractive fermion-fermion interaction.

Uniform Fermi superfluid (II)

The ground-state average of the number of fermions reads

$$N = \int d^3\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \langle\, \hat{\psi}^+_\sigma(\mathbf{r})\, \hat{\psi}_\sigma(\mathbf{r})\,\rangle \; . \qquad (3)$$

This total number N is fixed by the chemical potential μ which appears in Eq. (2).
In a Fermi system the largest eigenvalue of the two-body density matrix gives the **number of Cooper pairs**, which is half of the number of condensed fermions N_0. Thus one finds[3]

$$N_0 = 2 \int d^3\mathbf{r}_1 \; d^3\mathbf{r}_2 \; |\langle\, \hat{\psi}_\downarrow(\mathbf{r}_1)\; \hat{\psi}_\uparrow(\mathbf{r}_2)\,\rangle|^2 \; . \qquad (4)$$

[3] A.J. Leggett, Quantum liquids. Bose condensation and Cooper pairing in condensed-matter systems (Oxford Univ. Press, Oxford, 2006)

Uniform Fermi superfluid (III)

Within the Bogoliubov approach the mean-field Hamiltonian derived from Eq. (2) can be diagonalized by using the Bogoliubov-Valatin representation of the field operator $\hat{\psi}_\sigma(\mathbf{r})$ in terms of the anticommuting quasi-particle Bogoliubov operators $\hat{b}_{\mathbf{k}\sigma}$ with amplitudes $u_\mathbf{k}$ and $v_\mathbf{k}$ and the quasi-particle energy $E_\mathbf{k}$. In this way one finds familiar expressions for these quantities:

$$E_\mathbf{k} = \left[(\epsilon_\mathbf{k} - \mu)^2 + \Delta^2\right]^{1/2} \tag{5}$$

and

$$u_\mathbf{k}^2 = \left(1 + (\epsilon_\mathbf{k} - \mu)/E_\mathbf{k}\right)/2 \tag{6}$$

$$v_\mathbf{k}^2 = \left(1 - (\epsilon_\mathbf{k} - \mu)/E_\mathbf{k}\right)/2 , \tag{7}$$

where $\epsilon_\mathbf{k} = \hbar^2 k^2/(2m)$ is the single-particle energy.

Uniform Fermi superfluid (IV)

The parameter Δ is the pairing gap, which satisfies the gap equation

$$-\frac{1}{g} = \frac{1}{\Omega} \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}}} \, , \tag{8}$$

where Ω is the volume of the uniform system. Notice that this equation is **ultraviolet divergent** and it must be regularized.

The number equation, i.e. the equation for the total number density $n = N/\Omega$ of fermions is obtained from Eq. (3) as

$$n = \frac{2}{\Omega} \sum_{\mathbf{k}} v_{\mathbf{k}}^2 \, . \tag{9}$$

Finally, from Eq. (4) one finds that the condensate density $n_0 = N_0/\Omega$ of paired fermions is given by[4]

$$n_0 = \frac{2}{\Omega} \sum_{\mathbf{k}} u_{\mathbf{k}}^2 v_{\mathbf{k}}^2 \, . \tag{10}$$

[4] L.S., N. Manini, A. Parola, PRA **72**, 023621 (2005).

Uniform Fermi superfluid (V)

In three dimensions, a suitable regularization[5] of the gap equation is obtained by introducing the s-wave scattering length a via the equation

$$-\frac{1}{g} = -\frac{m}{4\pi\hbar^2 a} + \frac{1}{\Omega}\sum_{\mathbf{k}} \frac{m}{\hbar^2 k^2} \,, \tag{11}$$

and then subtracting this equation from the gap equation (8). In this way one obtains the three-dimensional **regularized gap equation**

$$-\frac{m}{4\pi\hbar^2 a} = \frac{1}{\Omega}\sum_{\mathbf{k}} \left(\frac{1}{2E_k} - \frac{m}{\hbar^2 k^2} \right) , \tag{12}$$

which can be used to study the full BCS-BEC crossover[6] by changing the amplitude and sign of the s-wave scattering length a.

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[5] Marini, Pistolesi, and Strinati, Eur. Phys. J. B **1**, 151 (1998).

[6] D.M. Eagles, PR **186**, 456 (1969); A.J. Leggett, in *Modern Trends in the Theory of Condensed Matter*, p. 13, edited by A. Pekalski and J. Przystawa (Springer, Berlin, 1980).

Uniform Fermi superfluid (VI)

Taking into account the functional dependence of the amplitudes u_k and v_k on μ and Δ, one finds[7] the condensate density

$$n_0 = \frac{m^{3/2}}{8\pi\hbar^3}\,\Delta^{3/2}\sqrt{\frac{\mu}{\Delta}+\sqrt{1+\frac{\mu^2}{\Delta^2}}}\,. \tag{13}$$

By the same techniques, also the two BCS-BEC equations can be written in a more compact form as

$$-\frac{1}{a} = \frac{2(2m)^{1/2}}{\pi\hbar^3}\,\Delta^{1/2}\,I_1\left(\frac{\mu}{\Delta}\right)\,, \tag{14}$$

$$n = \frac{(2m)^{3/2}}{2\pi^2\hbar^3}\,\Delta^{3/2}\,I_2\left(\frac{\mu}{\Delta}\right)\,, \tag{15}$$

where $I_1(x)$ and $I_2(x)$ are two monotonic functions which can be expressed in terms of elliptic integrals[8].

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[7] L.S., N. Manini, A. Parola, PRA **72**, 023621 (2005).

[8] Marini, Pistolesi, and Strinati, Eur. Phys. J. B **1**, 151 (1998).

Uniform Fermi superfluid (VII)

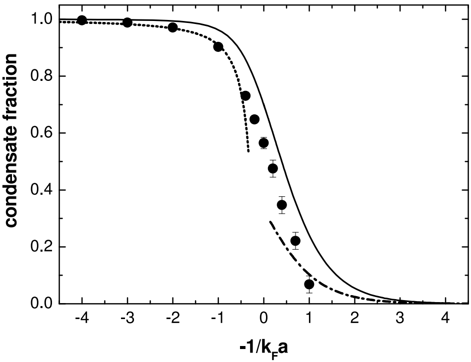

Figure: Condensate fraction of pairs as a function of the inverse interaction strength $y = 1/(k_F a)$: **our mean-field theory** (solid line); Fixed-Node Diffusion Monte Carlo results (symbols) [G. E. Astrakharchik et al., PRL **95**, 230405 (2005)]; Bogoliubov quantum depletion of a Bose gas with $a_m = 0.6a$ (dashed line); BCS theory (dot-dashed line).

Trapped Fermi superfluid (I)

In order to compare our theoretical condensate fraction with the MIT experiment[9] done with **trapped ^{6}Li atoms**, we use the local density approximation (LDA), given by

$$\mu \to \mu - U(\mathbf{r}) \tag{16}$$

where

$$U(\mathbf{r}) = \frac{m}{2}\left[\omega_\perp^2 (x^2 + y^2) + \omega_z^2 z^2\right] \tag{17}$$

is the external trapping potential.

In this way the energy gap $\Delta(\mathbf{r})$, the total density $n(\mathbf{r})$ and the condensate density $n_0(\mathbf{r})$ become local scalar fields.

[9] M.W. Zwierlein *et al.*, PRL **92**, 120403 (2004); M.W. Zwierlein *et al.*, PRL **94**, 180401 (2005).

Trapped Fermi superfluid (II)

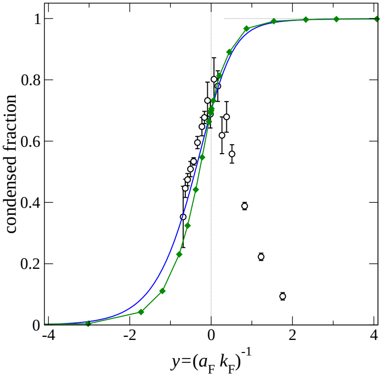

Figure: Solid line: condensate fraction $N_0/(N/2)$ of uniform Fermi superfluid vs $y = (k_F a_F)^{-1}$. Joined diamonds: same quantity for a droplet of $N = 6 \times 10^6$ fermions in harmonic trap, as in the MIT experiment, plotted against the value of y at the center of the trap. Open circles with error bars: condensed fraction of MIT experiment (inelastic losses?). [L.S., N. Manini, A. Parola, PRA **72**, 023621 (2005)].

Polarized Fermi superfluid (I)

Let us now analyze the polarized uniform two-spin-component Fermi superfluid made of ultracold atoms.
The shifted Hamiltonian is given by

$$\begin{aligned} \hat{H}' &= \int d^3\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^+_\sigma(\mathbf{r}) \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu_\sigma \right) \hat{\psi}_\sigma(\mathbf{r}) \\ &+ g\, \hat{\psi}^+_\uparrow(\mathbf{r})\, \hat{\psi}^+_\downarrow(\mathbf{r})\, \hat{\psi}_\downarrow(\mathbf{r})\, \hat{\psi}_\uparrow(\mathbf{r}) , \end{aligned} \tag{18}$$

where μ_σ is the chemical potential of each spin component.
We define the average chemical potential μ and the imbalance chemical potential ζ as

$$\mu = \frac{\mu_\uparrow + \mu_\downarrow}{2} \qquad \zeta = \frac{\mu_\uparrow - \mu_\downarrow}{2} . \tag{19}$$

Polarized Fermi superfluid (II)

We extend our previous mean-field results and find that the condensate number reads

$$N_0 = \sum_{|\mathbf{k}| \notin [k_-, k_+]} \frac{\Delta_0^2}{4E_k^2} \tag{20}$$

where $\frac{\hbar^2 k_\mp^2}{2m} = \max(\mu \mp \sqrt{\zeta^2 - \Delta_0^2}, 0)$ and $E_k = \sqrt{(\frac{\hbar^2 k^2}{2m} - \mu)^2 + \Delta_0^2}$.
We calculate the condensate fraction $\phi = N_0/N$ as a function of the **dimensionless interaction parameter**

$$y = \frac{1}{k_F a_s} , \tag{21}$$

with $k_F = (3\pi^2 n)^{1/3}$ the Fermi wavenumber, and of the **polarization**

$$P = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow} . \tag{22}$$

Polarized Fermi superfluid (III)

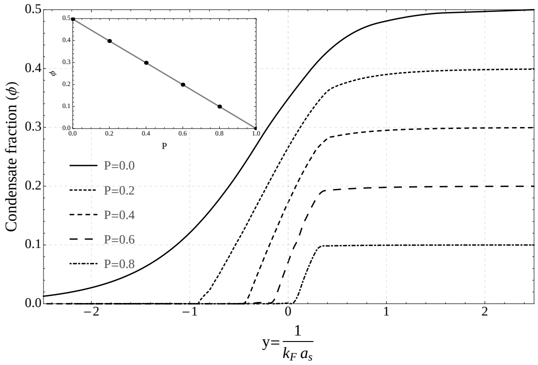

Figure: The condensate fraction $\phi = N_0/N$ as a function of the inverse dimensionless interaction parameter $y = 1/(k_F a_s)$ for different values of the polarization P. In the inset ϕ as a function of the polarization $P = (N_\uparrow - N_\downarrow)/(N_\uparrow + N_\downarrow)$ for $y = 2$. [G. Bighin, LS, G. Mazzarella, L. Dell'Anna, arXiv:1404.3178].

Polarized Fermi superfluid (IV)

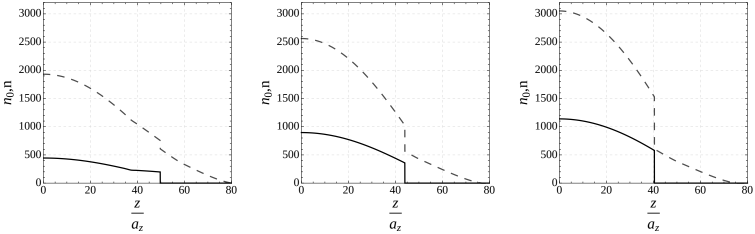

Figure: Polarized Fermi gas in the harmonic trap. The condensate density profile $n_0(z)$ (solid line) and total density profile $n(z)$ (dashed line) in the axial direction z. From left to right: $y = -0.44$, $y = 0.0$, $y = 0.11$. Number of atoms $N = 2.3 \cdot 10^7$ and polarization $P = (N_\uparrow - N_\downarrow)/N = 0.2$. [G. Bighin, LS, G. Mazzarella, L. Dell'Anna, arXiv:1404.3178].

Polarized Fermi superfluid (V)

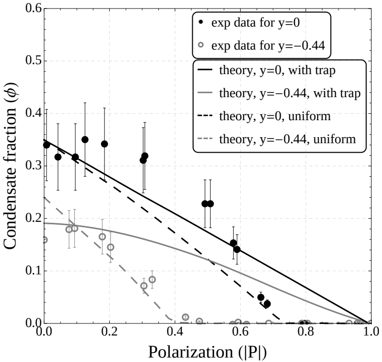

Figure: **Polarized and trapped Fermi superfluid**. Condensate fraction ϕ vs polarization $|P|$ for two values of $y = 1/k_F a_s$: $y = -0.44$ (open circles) and $y = 0.0$ (filled circles). Circles with error bars are experimental data of ^{6}Li atoms taken from MIT experiment (again inelastic losses?).[11] [G. Bighin, LS, G. Mazzarella, L. Dell'Anna, arXiv:1404.3178].

[10]M. Zwierlein, A. Schirotzek, C.H. Schunck, and W. Ketterle, Science **311**, 492 (2006).

Open problems

- Analysis of the condensate fraction for 2D systems[12]
- Condensate fraction in other systems: for instance neutron matter[13]
- Inclusion of spin-orbit terms[14]
- Condensate fraction with narrow resonances[15]
- Calculation of the condensate fraction beyond mean-field
- Finite-temperature effects on the condensate fraction

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[12] L.S., PRA **76**, 015601 (2007).
[13] L.S, PRC **84**, 067301 (2011).
[14] L. Dell'Anna, G. Mazzarella, and L.S., PRA **84**, 033633 (2011); PRA **86**, 053632 (2012).
[15] L.S, PRA **86**, 055602 (2012).

Acknowledgments

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