Resonant Fermi gas of atoms with spin-orbit coupling

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Summary

- BCS-BEC crossover
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In 2004 the BCS-BEC crossover has been observed with ultracold gases made of fermionic $^{40}$K and $^{6}$Li alkali-metal atoms.\(^1\)

This crossover is obtained by changing (with a Feshbach resonance) the s-wave scattering length $a_s$ of the inter-atomic potential:
- $a_s \rightarrow 0^-$ (BCS regime of weakly-interacting Cooper pairs)
- $a_s \rightarrow \pm \infty$ (unitarity limit of strongly-interacting Cooper pairs)
- $a_s \rightarrow 0^+$ (BEC regime of bosonic dimers)

\(^1\)C.A. Regal et al., PRL 92, 040403 (2004); M.W. Zwierlein et al., PRL 92, 120403 (2004); M. Bartenstein, A. Altmeyer et al., PRL 92, 120401 (2004); J. Kinast et al., PRL 92, 150402 (2004).
The crossover from a BCS superfluid ($a_s < 0$) to a BEC of molecular pairs ($a_s > 0$) has been investigated experimentally around a Feshbach resonance, where the s-wave scattering length $a_s$ diverges, and it has been shown that the system is (meta)stable. The detection of quantized vortices under rotation\(^2\) has clarified that this dilute and ultracold gas of Fermi atoms is superfluid. Usually the BCS-BEC crossover is analyzed in terms of

\[ y = \frac{1}{k_F a_s} \quad (1) \]

the inverse scaled interaction strength, where $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wave number and $n$ the total density. The system is dilute because $r_e k_F \ll 1$, with $r_e$ the effective range of the inter-atomic potential.

In 2011 and 2012 artificial spin-orbit coupling has been imposed on both bosonic\(^3\) and fermionic\(^4\) atomic gases. The single-particle Hamiltonian \(\hat{h}_{sp}\) with both Rashba and Dresselhaus spin-orbit couplings reads

\[
\hat{h}_{sp} = \frac{\hat{p}^2}{2m} + v_R (\hat{\sigma}_1 \hat{p}_y - \hat{\sigma}_2 \hat{p}_x) + v_D (\hat{\sigma}_1 \hat{p}_y + \hat{\sigma}_2 \hat{p}_x),
\]

with \(\hat{p}^2 = -\hbar^2 \nabla^2\), \(\hat{p}_x = -i\hbar \frac{\partial}{\partial x}\), \(\hat{p}_y = -i\hbar \frac{\partial}{\partial y}\), \(v_R\) and \(v_D\) the Rashba and Dresselhaus coupling constant, respectively, and

\[
\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.
\]


\(^4\)P. Wang et al., PRL 109, 095301 (2012); L.W. Cheuk et al., PRL 109, 095302 (2012).
The partition function $Z$ of the uniform two-spin-component Fermi system at temperature $T$, in a volume $V$, and with chemical potential $\mu$ can be written in terms of a functional integral as

$$Z = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{1}{\hbar} S \right\},$$

where

$$S = \int_0^{\hbar \beta} d\tau \int_V d^3r \, \mathcal{L}$$

is the Euclidean action functional and $\mathcal{L}$ is the Euclidean Lagrangian density, given by

$$\mathcal{L} = (\bar{\psi}_\uparrow, \bar{\psi}_\downarrow) \left[ \hbar \partial_\tau + \hbar_{sp} - \mu \right] \left( \begin{array}{c} \psi_\uparrow \\ \psi_\downarrow \end{array} \right) + g \, \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow$$

with $g$ is the strength of the s-wave coupling ($g < 0$ in the BCS regime). Notice that $\beta = 1/(k_B T)$ with $k_B$ the Boltzmann constant. In the rest of the seminar we shall use units such that $\hbar = m = k_B = 1$. 


The Lagrangian density $\mathcal{L}$ is quartic in the fermionic fields $\psi_s$, but one can reduce the problem to a quadratic Lagrangian density by introducing an auxiliary complex scalar field $\Delta(r, \tau)$ via Hubbard-Stratonovich transformation\(^5\), which gives

$$ Z = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \mathcal{D}[\Delta, \bar{\Delta}] \exp \{-S_e\}, \quad (6) $$

where

$$ S_e = \int_0^{1/T} d\tau \int_V d^3r \mathcal{L}_e \quad (7) $$

and the (exact) effective Euclidean Lagrangian density $\mathcal{L}_e$ reads

$$ \mathcal{L}_e = (\bar{\psi}_\uparrow, \bar{\psi}_\downarrow) \left[ \partial_\tau + \hat{h}_{sp} - \mu \right] \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} + \bar{\Delta} \psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow - \frac{|\Delta|^2}{g}. \quad (8) $$

Mean-field approach (III)

It is a standard procedure to integrate out the quadratic fermionic fields and to get a new formally-exact effective action $S_{\text{eff}}$ which depends only on the auxiliary field $\Delta(r, \tau)$. In this way we obtain

$$Z = \int \mathcal{D}[\Delta, \bar{\Delta}] \exp \{-S_{\text{eff}}\},$$

where

$$S_{\text{eff}} = -Tr[\ln (G^{-1})] - \int_{0}^{1/T} d\tau \int_{V} d^{3}r \frac{|\Delta|^{2}}{g}$$

with $\gamma(\hat{p}) = v_{R}(\hat{p}_{y} + i\hat{p}_{x}) + v_{D}(\hat{p}_{y} - i\hat{p}_{x})$ and

$$G^{-1} = \begin{pmatrix}
\partial_{\tau} + \frac{\hat{p}^{2}}{2m} - \mu & \Delta & \gamma(\hat{p}) & 0 \\
\Delta & \partial_{\tau} - \frac{\hat{p}^{2}}{2m} + \mu & 0 & -\gamma(-\hat{p}) \\
\bar{\gamma}(\hat{p}) & 0 & \partial_{\tau} + \frac{\hat{p}^{2}}{2m} - \mu & \Delta \\
0 & -\bar{\gamma}(-\hat{p}) & \Delta & \partial_{\tau} - \frac{\hat{p}^{2}}{2m} + \mu
\end{pmatrix}$$
Mean-field approach (IV)

For a uniform Fermi superfluid within the simplest mean-field approximation one has a constant and real gap parameter, i.e. \( \Delta(r, \tau) = \Delta \), and the partition function becomes\(^6\)

\[
Z_{mf} = \exp \left\{ -S_{mf} \right\} = \exp \left\{ -\frac{\Omega_{mf}}{T} \right\}, \quad (12)
\]

where

\[
S_{mf} = \frac{\Omega_{mf}}{T} = -\sum_k \left[ \sum_{j=1}^{4} \ln \left( 1 + e^{-E_{k,j}/T} \right) - \frac{\xi_k}{T} \right] - \frac{V}{T} \frac{\Delta^2}{g} \quad (13)
\]

with \( \xi_k = \hbar^2 k^2 / (2m) - \mu \), \( \gamma_k = \hbar \nu_R (k_y + ik_x) + \hbar \nu_D (k_y - ik_x) \), and

\[
E_{k,1} = \sqrt{\xi_k - |\gamma_k|^2 + \Delta^2}, \quad E_{k,3} = -E_{k,1}, \quad (14)
\]
\[
E_{k,2} = \sqrt{(\xi_k + |\gamma_k|^2 + \Delta^2}, \quad E_{k,4} = -E_{k,2}. \quad (15)
\]

\(^6\)L. Dell’Anna, G. Mazzarella, L.S., PRA 84, 033633 (2011).
Gap and number equations (I)

The constant and real gap parameter $\Delta$ is obtained from

$$\frac{\partial S_{mf}}{\partial \Delta} = 0 ,$$  \hspace{1cm} (16)

which gives the **gap equation**

$$- \frac{1}{g} = \frac{1}{V} \sum_k \sum_{j=1,2} \frac{\tanh (E_{k,j}/2T)}{4E_{k,j}} .$$  \hspace{1cm} (17)

The integral on the right side of this equation is formally divergent. However, expressing the bare interaction strength $g$ in terms of the physical scattering length $a_s$ with the formula$^7$

$$- \frac{1}{g} = - \frac{1}{4\pi a_s} + \frac{1}{V} \sum_k \frac{1}{k^2}$$  \hspace{1cm} (18)

one obtains the **regularized gap equation$^8$**

$$- \frac{1}{4\pi a_s} = \frac{1}{V} \sum_k \left[ \sum_{j=1,2} \frac{\tanh (E_{k,j}/2T)}{4E_{k,j}} - \frac{1}{k^2} \right] .$$  \hspace{1cm} (19)

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From the thermodynamic formula

\[ N = - \left( \frac{\partial \Omega_{mf}}{\partial \mu} \right)_{V,T} \]  

(20)

one obtains also the equation for the number of particles\(^9\)

\[ N = \sum_k \left( 1 - \frac{\xi_k - |\gamma_k|}{2E_{k,1}} \tanh \left( \frac{E_{k,1}}{2T} \right) - \frac{\xi_k + |\gamma_k|}{2E_{k,2}} \tanh \left( \frac{E_{k,2}}{2T} \right) \right) . \]  

(21)

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Singlet and triplet condensation (I)

In a Fermi system the largest eigenvalue $N_C$ of the two-body density matrix gives the number of fermion pairs which have their center of mass with zero linear momentum.\textsuperscript{10} This condensed number of pairs is given by

$$N_C = N_0 + N_1 ,$$

where

$$N_0 = 2 \int d^3r \, d^3r' \, |\langle \psi_\downarrow(r) \, \psi_\uparrow(r') \rangle|^2 ,$$

is the condensed number of pairs in the singlet state (spin 0), while

$$N_1 = 2 \int d^3r \, d^3r' \, |\langle \psi_\uparrow(r) \, \psi_\uparrow(r') \rangle|^2 .$$

is the condensed number of pairs in the triplet state (spin 1).

Singlet and triplet condensation (II)

In our superfluid Fermi system with spin-orbit coupling we obtain\textsuperscript{11}

\[ N_0 = \frac{\Delta^2}{4} \sum_k \left( \frac{1}{2E_{k,1}} \tanh \left( \frac{E_{k,1}}{2T} \right) + \frac{1}{2E_{k,2}} \tanh \left( \frac{E_{k,2}}{2T} \right) \right)^2 . \]  \hspace{1cm} (25)

and

\[ N_1 = \frac{\Delta^2}{4} \sum_k \left( \frac{1}{2E_{k,1}} \tanh \left( \frac{E_{k,1}}{2T} \right) - \frac{1}{2E_{k,2}} \tanh \left( \frac{E_{k,2}}{2T} \right) \right)^2 . \]  \hspace{1cm} (26)

Notice that in the absence of spin-orbit coupling \((v_R = v_D = 0)\) one has \(E_{k,1} = E_{k,2}\) and consequently the condensate number \(N_1\) of Cooper pairs in the triplet state is zero.

\textsuperscript{11}L. Dell’Anna, G. Mazzarella, L.S., PRA 84, 033633 (2011).
Results with Rashba coupling at $T = 0$ (I)

We are interested in the low temperature regime where the condensate fraction can be quite large. Quantitatively we restrict our study to the zero temperature limit ($T=0$). In the equations above we have therefore simply $\tanh(E_{k,j}/2T) \rightarrow 1$.

In this way the regularized gap equation is given by

$$\frac{1}{4\pi a_s} = \frac{1}{V} \sum_k \left[ \sum_{j=1,2} \frac{1}{4E_{k,j}} - \frac{1}{k^2} \right],$$

while the number equation reads

$$N = \sum_k \left( 1 - \frac{\xi_k - |\gamma_k|}{2E_{k,1}} - \frac{\xi_k + |\gamma_k|}{2E_{k,2}} \right).$$
Similarly, we obtain for the singlet condensate number

\[ N_0 = \frac{\Delta^2}{4} \sum_k \left( \frac{1}{2E_{k,1}} \pm \frac{1}{2E_{k,2}} \right)^2. \]  

(29)

and for the triplet condensate number

\[ N_1 = \frac{\Delta^2}{4} \sum_k \left( \frac{1}{2E_{k,1}} - \frac{1}{2E_{k,2}} \right)^2. \]  

(30)

From the previous equations one can calculate the chemical potential \( \mu \), the energy gap \( \Delta \), and also the condensate fractions \( N_0/N \) and \( N_1/N \), as a function of the scaled interaction strength \( y = 1/(k_F a_S) \).

For simplicity, we show the results obtained for \( \nu_D = 0 \), i.e. when only Rashba spin-orbit coupling is active.
Results with Rashba coupling at $T = 0$ (III)

Scaled chemical potential $\mu/\epsilon_F$ as a function of the adimensional interaction strength $y = 1/(k_F a_s)$ for different values of the scaled Rashba velocity: $v_R/v_F = 0$ (solid line), $v_R/v_F = 0.7$ (long-dashed line), $v_R/v_F = 1$ (short-dashed line), $v_R/v_F = 1.4$ (dotted line), $v_R/v_F = 2$ (dashed-dotted line). Here $\epsilon_F = v_F^2/2$ is the Fermi energy and $v_F = (3\pi^2 n)^{1/3}$ is the Fermi velocity.
Results with Rashba coupling at $T = 0$ (VI)

Scaled energy gap $\Delta/\epsilon_F$ as a function of the adimensional interaction strength $y = 1/(k_F a_s)$ for different values of the scaled Rashba velocity: $v_R/v_F = 0$ (solid line), $v_R/v_F = 0.7$ (long-dashed line), $v_R/v_F = 1$ (short-dashed line), $v_R/v_F = 1.4$ (dotted line), $v_R/v_F = 2$ (dashed-dotted line). Here $\epsilon_F = v_F^2/2$ is the Fermi energy and $v_F = (3\pi^2 n)^{1/3}$ is the Fermi velocity.
Results with Rashba coupling at $T = 0$ (V)

Singlet condensate fraction $N_0/N$ (upper curves) and triplet condensate fraction $N_1/N$ (lower curves) as a function of the adimensional interaction strength $y = 1/(k_F a_s)$ for different values of the scaled Rashba velocity: $v_R/v_F = 0$ (solid line), $v_R/v_F = 0.7$ (long-dashed line), $v_R/v_F = 1$ (short-dashed line), $v_R/v_F = 1.4$ (dotted line), $v_R/v_F = 2$ (dashed-dotted line). Here $v_F = (3\pi^2 n)^{1/3}$ is the Fermi velocity.
Conclusions

Unlike the chemical potential $\mu$ and the pairing gap $\Delta$ which exhibit no particular behavior at the crossover, the condensate fraction is very peculiar.

The condensation of singlet pairs ($N_0/N$) is promoted by Rashba coupling in the BCS regime whereas it is suppressed in the BEC regime.

The triplet contribution $N_1/N$ to the condensate fraction has not a monotonic behavior as a function of the scattering parameter, becoming larger close to the crossover.

In a recent paper\textsuperscript{12} we have shown that by including also the Dresselhaus spin-orbit coupling the singlet condensate fraction simply decreases, while the triplet condensate fraction is suppressed in the BCS regime and increased in the BEC regime.

\textsuperscript{12}L. Dell’Anna, G. Mazzarella, L.S., PRA \textbf{86}, 053632 (2012).
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