Shock waves in the unitary Fermi gas

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Banff, May 2015

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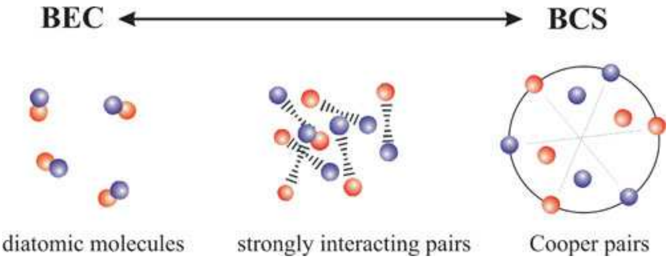
Summary

- 1. BCS-BEC crossover and the unitarity limit
- 2. Thomas-Fermi density functional
- 3. Extended Thomas-Fermi density functional
- 4. Extended superfluid hydrodynamics
- 5. Shock waves perturbing the uniform Fermi gas
- 6. Shock waves in colliding Fermi clouds
- Conclusions

1. BCS-BEC crossover and the unitarity limit (I)

In 2002 the BCS-BEC crossover has been observed[1] with ultracold gases made of fermionic alkali-metal atoms.



This crossover is obtained by changing (with a Feshbach resonance) the s-wave scattering length a_F of the inter-atomic potential:
– $a_F \to 0^-$ (BCS regime of weakly-interacting Cooper pairs)
– $a_F \to \pm\infty$ (unitarity limit of strongly-interacting Cooper pairs)
– $a_F \to 0^+$ (BEC regime of bosonic dimers)

[1] K.M. O'Hara *et al.*, Science **298**, 2179 (2002).

1. BCS-BEC crossover and the unitarity limit (II)

The many-body Hamiltonian of a two-spin-component Fermi system is given by

$$\hat{H} = \sum_{i=1}^{N_\uparrow} \left(\frac{\hat{p}_i^2}{2m} + U(\mathbf{r}_i) \right) + \sum_{j=1}^{N_\downarrow} \left(\frac{\hat{p}_j^2}{2m} + U(\mathbf{r}_j) \right) + \sum_{i,j} V(\mathbf{r}_i - \mathbf{r}_j) \ , \quad (1)$$

where $U(\mathbf{r})$ is the external confining potential and $V(\mathbf{r})$ is the inter-atomic potential. Here we consider $N_\uparrow = N_\downarrow$.
The inter-atomic potential of a dilute gas can be modelled by a square well potential:

$$V(r) = \begin{cases} -V_0 & r < r_0 \\ 0 & r > r_0 \end{cases} \quad (2)$$

By varying the depth V_0 of the potential one changes the s-wave scattering length

$$a_F = r_0 \left(1 - \frac{tan(r_0\sqrt{mV_0}/\hbar)}{r_0\sqrt{mV_0}/\hbar} \right) \ . \quad (3)$$

1. BCS-BEC crossover and the unitarity limit (III)

The crossover from a BCS superfluid ($a_F < 0$) to a BEC of molecular pairs ($a_F > 0$) has been investigated experimentally[2], and it has been shown that the unitary Fermi gas ($|a_F| = \infty$) exists and is (meta)stable. In few words, the unitarity regime of a dilute Fermi gas is characterized by

$$r_0 \ll n^{-1/3} \ll |a_F| \; . \tag{4}$$

Under these conditions the Fermi gas is called unitary Fermi gas. Ideally, the unitarity limit corresponds to

$$r_0 = 0 \qquad \text{and} \qquad a_F = \pm\infty \; . \tag{5}$$

The detection of quantized vortices under rotation[3] has clarified that the unitary Fermi gas is superfluid.

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[2] K.M. O'Hara *et al.*, Science **298**, 2179 (2002).

[3] M.W. Zwierlein *et al.*, Science **311**, 492 (2006); M.W. Zwierlein *et al.*, Nature **442**, 54 (2006)

1. BCS-BEC crossover and the unitarity limit (IV)

The only length characterizing the uniform unitary Fermi gas is the average distance between particles $d = n^{-1/3}$.
In this case, from simple dimensional arguments, the ground-state energy per volume must be

$$\frac{E_0}{V} = \xi \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n^{5/3} = \xi \; \frac{3}{5} \; \epsilon_F \; n \; , \tag{6}$$

with ϵ_F Fermi energy of the ideal gas, $n = N/V$ the total density, and ξ a universal unknown parameter.
Monte Carlo calculations and experimental data with dilute and ultracold atoms suggest[4] that the unitary Fermi gas is a superfluid with $\xi \simeq 0.4$.

[4]S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008).

2. Thomas-Fermi density functional

The Thomas-Fermi (TF) energy functional[5] of the unitary Fermi gas in an external potential $U(\mathbf{r})$ is

$$E_{TF} = \int d^3\mathbf{r} \left[\xi \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n^{5/3}(\mathbf{r}) + U(\mathbf{r}) n(\mathbf{r}) \right] , \qquad (7)$$

with $n(\mathbf{r}) = n_\uparrow(\mathbf{r}) + n_\downarrow(\mathbf{r})$ total local density. The total number of fermions is

$$N = \int d^3\mathbf{r} \; n(\mathbf{r}) . \qquad (8)$$

By minimizing E_{TF} one finds

$$\xi \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n^{2/3}(\mathbf{r}) + U(\mathbf{r}) = \bar{\mu} , \qquad (9)$$

with $\bar{\mu}$ chemical potential of the non uniform system.

[5] S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008).

The TF functional must be extended to cure the pathological TF behavior at the surface. We add to the energy per particle the gradient term **[LS and F. Toigo, Phys. Rev. A 78, 053626 (2008)]**

$$\lambda \frac{\hbar^2}{8m} \frac{(\nabla n)^2}{n^2} = \lambda \frac{\hbar^2}{2m} \frac{(\nabla \sqrt{n})^2}{n} \ . \qquad (10)$$

Historically, this term was introduced by von Weizsäcker[6] to treat surface effects in nuclei. Here we consider λ as a phenomenological parameter accounting for the increase of kinetic energy due the spatial variation of the density.
Other recent density-functional methods for unitary Fermi gas:
– the Kohn-Sham density functional approach of Papenbrock, Phys. Rev. A **72**, 041603 (2005);
– the superfluid local-density approximation of Bulgac, Phys. Rev. A **76**, 040502(R) (2007).

[6] C.F. von Weizsäcker, Zeit. Phys. **96**, 431 (1935).

3. Extended Thomas-Fermi density functional (II)

The new energy functional, that is the extended Thomas-Fermi (ETF) functional of the unitary Fermi gas, reads

$$E = \int d^3\mathbf{r} \left[\lambda \frac{\hbar^2}{8m} \frac{(\nabla n(\mathbf{r}))^2}{n(\mathbf{r})} + \xi \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{5/3} n(\mathbf{r})^{5/3} + U(\mathbf{r}) n(\mathbf{r}) \right] . \tag{11}$$

By minimizing the ETF energy functional one gets:

$$\left[\lambda \frac{\hbar^2}{2m} \nabla^2 + \xi \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n(\mathbf{r})^{2/3} + U(\mathbf{r}) \right] \sqrt{n(\mathbf{r})} = \bar{\mu} \sqrt{n(\mathbf{r})} . \tag{12}$$

This is a sort of stationary 3D nonlinear Schrödinger (3D NLS) equation. The simple and reasonable choice

$$\xi = 0.44 \qquad \text{and} \qquad \lambda = 1/4 \tag{13}$$

fits quite well Monte Carlo data.[7]

[7] Indeed $\lambda = 1/4$ is the best choice to describe the DC Josephson effect of the unitary Fermi superfluid: see F. Ancilotto, LS, and F. Toigo, Phys. Rev. A **79**, 033627 (2009).

3. Extended Thomas-Fermi density functional (III)

Having determined the parameters ξ and λ we can now use our single-orbital density functional to calculate various properties of the trapped unitary Fermi gas.
We calculate numerically (by solving with a finite-difference Crank-Nicolson method the stationary 3D NLSE) the density profile $n(\mathbf{r})$ of the gas in a isotropic harmonic trap

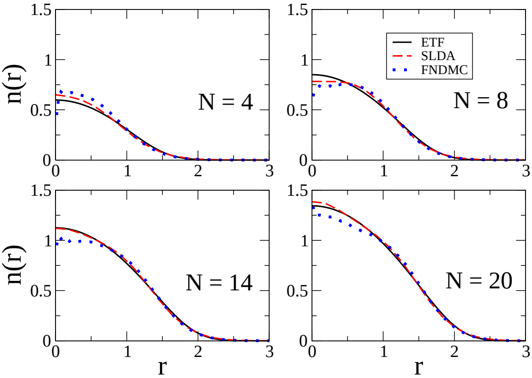
$$U(\mathbf{r}) = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2) \ . \tag{14}$$

We compare our results with those obtained by Doerte Blume[8] with her FNDMC code. For completeness we consider also the density profiles obtained by Aurel Bulgac[9] using his multi-orbital density functional (SLDA).

[8] D. Blume, J. von Stecher, C.H. Greene, Phys. Rev. Lett. **99**, 233201 (2007); J. von Stecher, C.H. Greene and D. Blume, Phys. Rev. A **77** 043619 (2008); D. Blume, unpublished.

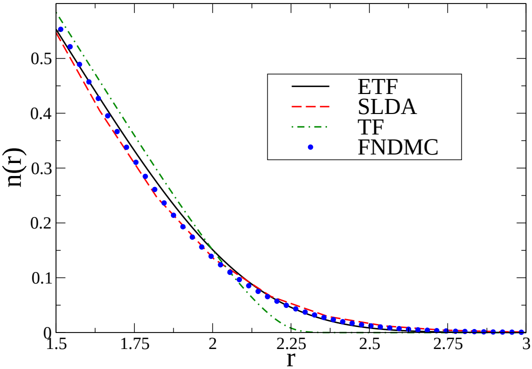
[9] A. Bulgac, Phys. Rev. A **76**, 040502(R) (2007).

3. Extended Thomas-Fermi density functional (IV)



Unitary Fermi gas under harmonic confinement of frequency ω. Density profiles $n(r)$ for N (even) fermions obtained with our ETF (solid lines), Bulgac's SLDA (dashed lines) and FNDMC (circles). Lengths in units of $a_H = \sqrt{\hbar/(m\omega)}$. [L.S., F. Ancilotto and F. Toigo, Laser Phys. Lett. **7**, 78 (2010).]

3. Extended Thomas-Fermi density functional (V)



Zoom of the density profile $n(r)$ for $N = 20$ fermions near the surface obtained with our ETF (solid lines), Bulgac's SLDA (circles) and FNDMC (circles). Lengths in units of $a_H = \sqrt{\hbar/(m\omega)}$. [L.S., F. Ancilotto and F. Toigo, Laser Phys. Lett. **7**, 78 (2010).]

4. Extended superfluid hydrodynamics (I)

Let us now analyze the effect of the gradient term on the dynamics of the superfluid unitary Fermi gas.

At zero temperature the low-energy collective dynamics of this fermionic gas can be described by the equations of extended[10] irrotational and inviscid hydrodynamics:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 \, , \quad (15)$$

$$m\frac{\partial}{\partial t}\mathbf{v} + \nabla\Big[-\lambda \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} + \xi \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} + U(\mathbf{r}) + \frac{m}{2}v^2 \Big] = 0 \, . \quad (16)$$

They are the simplest extension of the equations of superfluid hydrodynamics of fermions[11], where $\lambda = 0$.

[10] Quantum hydrodynamics of electrons: N. H. March and M. P. Tosi, Proc. R. Soc. A **330**, 373 (1972); E. Zaremba and H.C. Tso, Phys. Rev. B **49**, 8147 (1994).

[11] S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008).

4. Extended superfluid hydrodynamics (II)

The extended hydrodynamics equations can be written in terms of a superfluid nonlinear Schrödinger equation (superfluid NLSE).[12]
In fact, by introducing the complex wave function

$$\psi(\mathbf{r},t) = n(\mathbf{r},t)^{1/2}\ e^{i\theta(\mathbf{r},t)} \tag{17}$$

which is normalized to the total number N of superfuid atoms, and taking into account the correct phase-velocity relationship

$$\mathbf{v}(\mathbf{r},t) = \frac{\hbar^2}{2m}\nabla\theta(\mathbf{r},t)\ , \tag{18}$$

the equation

$$i\hbar\frac{\partial}{\partial t}\psi = [-\frac{\hbar^2}{4m}\nabla^2 + U(\mathbf{r}) + 2\mu + 2\xi\frac{\hbar^2}{2m}(3\pi^2|\psi|^2)^{2/3} + (1-4\lambda)\frac{\hbar^2}{4m}\frac{\nabla^2|\psi|}{|\psi|}]\psi \tag{19}$$

is equivalent to the equations of extended superfluid hydrodynamics.

[12] LS and F. Toigo, Phys. Rev. A **78**, 053626 (2008); LS, Laser Phys. **19**, 642 (2009); S.K. Adhikari and LS, New J. Phys **11**, 023011 (2009).

4. Extended superfluid hydrodynamics (III)

From the equations of superfluid hydrodynamics one finds the dispersion relation of low-energy collective modes of the uniform ($U(\mathbf{r}) = 0$) unitary Fermi gas in the form

$$\Omega_{col} = c_1 \ q \ , \tag{20}$$

where Ω_{col} is the collective frequency, q is the wave number and

$$c_1 = \sqrt{\frac{\xi}{3}} v_F \tag{21}$$

is the first sound velocity, with $v_F = \sqrt{\frac{2\epsilon_F}{m}}$ is the Fermi velocity of a noninteracting Fermi gas.

The equations of extended superfluid hydrodynamics (or the superfluid NLSE) give [**L.S. and F. Toigo, Phys. Rev. A 78, 053626 (2008)**] also a correcting term, i.e.

$$\Omega_{col} = c_1 \ q \ \sqrt{1 + \frac{3\lambda}{\xi} \left(\frac{\hbar q}{2 m v_F}\right)^2} \ , \tag{22}$$

which depends on the ratio λ/ξ.

5. Shock waves perturbing the uniform Fermi gas (I)

One of the basic problems in physics is how density perturbations propagate through a material.
In addition to the well-known sound waves, there are shock waves characterized by an abrupt change in the density of the medium: they produce, after a transient time, an extremely large density gradient (the shock).
Shock waves are ubiquitous and have been studied in many different physical systems[13]
Here we investigate the formation and dynamics of shock waves perturbing the uniform unitary Fermi gas by using the zero-temperature equations of generalized superfluid hydrodynamics.[14]

[13] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon Press, London, 1987); G.G. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[14] **LS, EPL 96, 40007 (2011)**

5. Shock waves perturbing the uniform Fermi gas (II)

To get analytical results we perform the following factorization

$$n(\mathbf{r}) = \bar{n}\,\rho(z) \tag{23}$$

where $\rho(z,t)$ is the relative density, i.e. the localized axial modification with respect to the uniform density $\bar{n}$.
Neglecting the gradient term we find the 1D hydrodynamic equations for the axial dynamics of the superfluid, given by

$$\dot{\rho} + v\rho' + v'\rho = 0\ , \tag{24}$$

$$\dot{v} + vv' + \frac{c_{ls}(\rho)^2}{\rho}\rho' = 0\ , \tag{25}$$

where dots denote time derivatives, primes space derivatives, and

$$c_{ls}(\rho) = c_s\rho^{1/3} \tag{26}$$

is the local sound velocity, with $c_s = c_{ls}(1) = \sqrt{\xi/3}v_F$ the bulk sound velocity, $v_F = \sqrt{2\epsilon_F/m}$ is bulk Fermi velocity and $\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2\bar{n})^{2/3}$ the bulk Fermi energy.

5. Shock waves perturbing the uniform Fermi gas (III)

The bulk sound velocity c_s is the speed of propagation of a small perturbation $\tilde{\rho}(z,t)$ with respect to the uniform superfluid of density $\bar{n}$. In fact, setting

$$\rho(z,t) = 1 + \tilde{\rho}(z,t) \tag{27}$$

with $\tilde{\rho}(z,t) \ll 1$ and $v(z,t)$ of the same order of $\tilde{\rho}(z,t)$, from the linearization of Eqs. (24) and (25) we get the familiar linear wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial z^2} \right) \tilde{\rho}(z,t) = 0 \ , \tag{28}$$

for $\tilde{\rho}(z,t)$ and a similar equation for $v(z,t)$. Modelling the initial perturbation with a Gaussian shape, i.e.

$$\tilde{\rho}(z,0) = 2\eta \ e^{-z^2/(2\sigma^2)} \ , \tag{29}$$

one finds from the linearized equations

$$\tilde{\rho}(z,t) = \eta \ e^{-(z-c_s t)^2/(2\sigma^2)} + \eta \ e^{-(z+c_s t)^2/(2\sigma^2)} \ , \tag{30}$$

with initial condition $\dot{\tilde{\rho}}(z,t=0) = 0$. Thus, for the conservation of the linear momentum, the initial wave packet splits into two (very small) waves travelling in opposite directions with the speed of sound c_s.

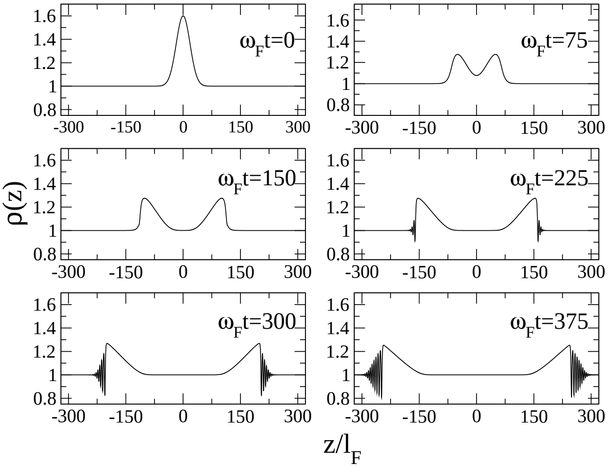
5. Shock waves perturbing the uniform Fermi gas (IV)

Obviously Eq. (30) is reliable only if $|\eta| \ll 1$. As expected, a small (infinitesimal) perturbation gives rise to sound waves.

What happens with a large (finite) perturbation?
a) The traveling waves are deformed: the local velocity depends on the local density.[15]
b) The gradient term cannot be neglected: it is essential during (and after) the formation of the shock.

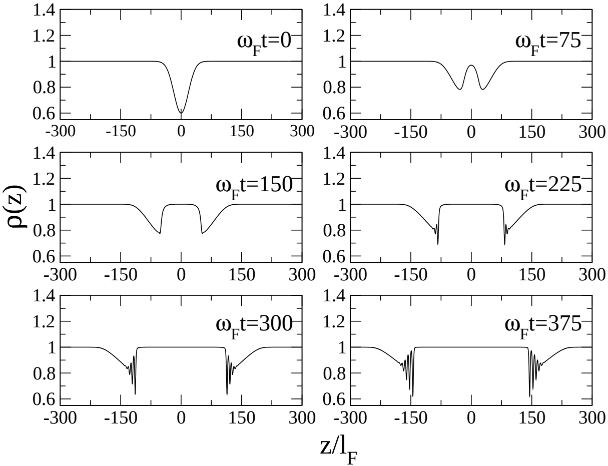
[15] Analytical details on the local velocity and the period of formation of shock waves can be found in **LS, EPL 96, 40007 (2011)**.

5. Shock waves perturbing the uniform Fermi gas (V)



Time evolution of supersonic shock waves. Initial condition with $\sigma/l_F = 18$ and $\eta = 0.3$. The curves give the relative density profile $\rho(z)$ at subsequent frames, where $l_F = \sqrt{\hbar^2/(m\epsilon_F)}$ is the Fermi length and $\omega_F = \epsilon_F/\hbar$ is the Fermi frequency. [LS, EPL **96**, 40007 (2011)]

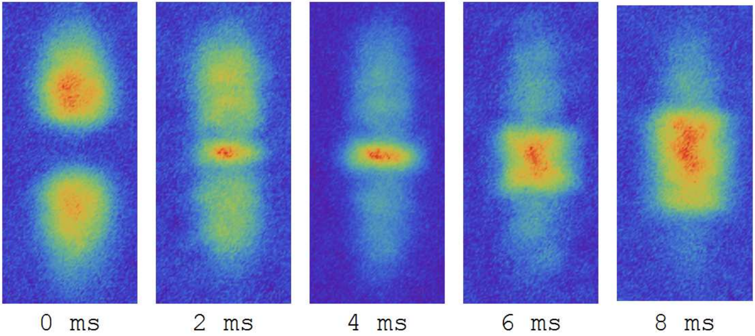
5. Shock waves perturbing the uniform Fermi gas (VI)



Time evolution of subsonic shock waves. Initial condition with $\sigma/l_F = 18$ and $\eta = -0.2$. The curves give the relative density profile $\rho(z)$ at subsequent frames, where $l_F = \sqrt{\hbar^2/(m\epsilon_F)}$ is the Fermi length and $\omega_F = \epsilon_F/\hbar$ is the Fermi frequency. [LS, EPL **96**, 40007 (2011)]

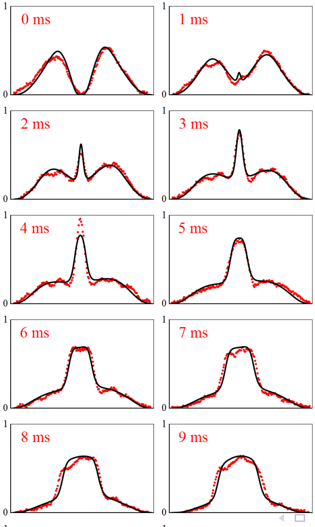
6. Shock waves in colliding Fermi clouds (I)

In 2011 the group of John Thomas (Duke and NCSU) made a remarkable experiment on the collision of two Fermi clouds of ^{6}Li atoms at unitarity containing about 10^5 atoms per spin [Phys. Rev. Lett. **106**, 150401 (2011)].



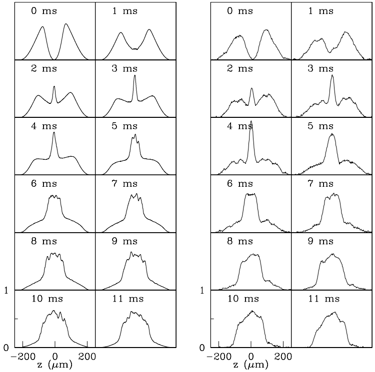
6. Shock waves in colliding Fermi clouds (II)

The experiment of Thomas was nicely reproduced using dissipative hydrodynamic equations with a very large fitting viscosity η [Phys. Rev. Lett. **106**, 150401 (2011)] which is however **unphysical**.



6. Shock waves in colliding Fermi clouds (III)

The experiment of Thomas is also nicely reproduced using our dispersive hydrodynamic equations (extended superfluid hydrodynamics) without fitting parameters [**F. Ancilotto, LS, and F. Toigo, Phys. Rev. A 85, 063612 (2012)**].



1D density profiles at different times t. Left part: our theory. Right part: experimental data. The normalized density is in units of $10^{-2}/\mu m$ per particle.

Conclusions

- Our extended Thomas-Fermi functional of the unitary Fermi gas can be used to study ground-state density profiles in a generic external potential $U(\mathbf{r})$.
- Our extended superfluid hydrodynamics, which are equivalent to a nonlinear Schrödinger equation, can be applied to investigate collective modes of the unitary Fermi gas.
- Also **shock waves** can be studied with our extended superfluid hydrodynamics and we suggest that that the **shock waves** observed in the collision of two unitary Fermi clouds are dispersive and not dissipative.