Bright solitons of attractive Bose-Einstein condensates confined in a quasi-1D optical lattice

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Summary

- BEC in a quasi-1D optical lattice
- Axial discretization of the 3D GPE
- Transverse dimensional reduction of the 3D DGPE
- 1D DNPSE
- Numerical results
- Collapse of the discrete bright soliton
- Conclusions
We consider a dilute Bose-Einstein condensate (BEC) confined in the $z$ direction by a **generic axial potential** $V(z)$ and in the plane $(x, y)$ by the **transverse harmonic potential**

$$U(x, y) = \frac{m}{2} \omega_\perp^2 (x^2 + y^2).$$

The characteristic harmonic length is given by

$$a_\perp = \sqrt{\frac{\hbar}{m\omega_\perp}},$$

and, for simplicity, we choose $a_\perp$ and $\omega_\perp^{-1}$, as length and time units, and $\hbar\omega_\perp$ as energy unit.
BEC in a quasi-1D optical lattice (II)

We assume that the system is well described by the 3D Gross-Pitaevskii equation (GPE), and in scaled units it reads

\[ i \frac{\partial}{\partial t} \psi(r, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} (x^2 + y^2) + V(z) + 2\pi g|\psi(r, t)|^2 \right] \psi(r, t), \]

where \( \psi(r, t) \) is the macroscopic wave function of the condensate normalized to the total number \( N \) of atoms and \( g = 2a_s/a_\perp \) with \( a_s \) the s-wave scattering length of the inter-atomic potential.

In addition, we suppose that the axial potential is the combination of periodic and harmonic potentials, i.e.

\[ V(z) = V_0 \cos(2kz) + \frac{1}{2} \lambda^2 z^2. \]

This potential models the quasi-1D optical lattice produced in experiments with Bose-Einstein condensates by using counter-propagating laser beams.\(^1\) Here \( \lambda = \omega_z/\omega_\perp \ll 1 \) models a weak axial harmonic confinement.

\(^1\)O. Morsch and M. Oberthaler, Rev. Mod. Phys. 78, 179 (2006).
Axial discretization of the 3D GPE (I)

We now perform a discretization of the 3D GPE along the $z$ axis due to the presence on the periodic potential. In particular we set

$$\psi(r, t) = \sum_n \phi_n(x, y, t) \ W_n(z) \quad (5)$$

where $W_n(z)$ is the **Wannier function** maximally localized at the $n$-th minimum of the axial periodic potential. This tight-binding ansatz is reliable in the case of a deep optical lattice.\(^2\)

Axial discretization of the 3D GPE (II)

We insert this ansatz into Eq. (3), multiply the resulting equation by $W_n^*(z)$ and integrate over $z$ variable. In this way we get

$$i \frac{\partial}{\partial t} \phi_n = \left[ -\frac{1}{2} \nabla^2_\perp + \frac{1}{2} (x^2 + y^2) + \epsilon_n \right] \phi_n - J (\phi_{n+1} + \phi_{n-1}) + 2\pi U |\phi_n|^2 \phi_n,$$

where the parameters $\epsilon$, $J$ and $U$ are given by

$$\epsilon_n = \int W_n^*(z) \left[ -\frac{1}{2} \frac{\partial^2}{\partial z^2} + V(z) \right] W_n(z) \, dz,$$

$$J = -\int W_{n+1}^*(z) \left[ -\frac{1}{2} \frac{\partial^2}{\partial z^2} + V(z) \right] W_n(z) \, dz,$$

$$U = g \int |W_n(z)|^4 \, dz.$$

The parameters $J$ and $U$ are practically independent on the site index $n$ and in the tight-binding regime $J > 0$. 
To further simplify the problem we set

$$\phi_n(x, y) = \frac{1}{\pi^{1/2}\sigma_n(t)} \exp \left[ - \left( \frac{x^2 + y^2}{2\sigma_n(t)^2} \right) \right] f_n(t), \quad (10)$$

where $\sigma_n(t)$ and $f_n(t)$, which account for **discrete transverse width** and **discrete axial wave function**, are the effective generalized coordinates to be determined variationally.

We insert this ansatz into the Lagrangian density associated to Eq. (6) and integrate over $x$ and $y$ variables. In this way we obtain an effective Lagrangian for the fields $f_n(t)$ and $\sigma_n(t)$.

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The Euler-Lagrange equation of the effective Lagrangian with respect to $f_n^*$ is
\[ i \frac{\partial}{\partial t} f_n = \left[ \frac{1}{2} \left( \frac{1}{\sigma_n^2} + \sigma_n^2 \right) + \epsilon_n \right] f_n - J (f_{n+1} + f_{n-1}) + \frac{U}{\sigma_n^2} |f_n|^2 f_n . \] (11)

while the Euler-Lagrange equation with respect to $\sigma_n$ gives
\[ \sigma_n^4 = 1 + U|f_n|^2 . \] (12)

Inserting Eq. (12) into Eq. (11) we finally get
\[ i \frac{\partial}{\partial t} f_n = \epsilon_n f_n - J (f_{n+1} + f_{n-1}) + \frac{1 + (3/2)U|f_n|^2}{\sqrt{1 + U|f_n|^2}} f_n , \] (13)

that is the 1D discrete nonpolynomial Schrödinger equation (DNPSE), describing the BEC under a transverse anisotropic harmonic confinement and an axial optical lattice.
1D DNPSE (II)

The 1D NPSE reduces to the familiar 1D DGPE (1D cubic DNLSE)

\[ i \frac{\partial}{\partial t} f_n = \epsilon_n f_n - J (f_{n+1} + f_{n-1}) + U|f_n|^2 f_n \]  \hspace{1cm} (14)

in the weak-coupling limit \(|U||f_n|^2 \ll 1\), where \(U\) can be both positive and negative. On the contrary, it becomes a 1D quadratic DNLSE

\[ i \frac{\partial}{\partial t} f_n = \epsilon_n f_n - J (f_{n+1} + f_{n-1}) + (3/2)\sqrt{U}|f_n|f_n \] \hspace{1cm} (15)

in the strong-coupling limit \(U|f_n|^2 \gg 1\), where \(U > 0\).
We have solved numerically both 1D DNPSE and 1D DGPE by using a Crank-Nicolson predictor-corrector algorithm with imaginary time to get the ground-state of the system.

In the next two slides we report our results obtained with $N = 100$ atoms in a quasi-1D optical lattice with weak axial harmonic confinement: $\lambda = \omega_z/\omega_\perp = 0.1$.

The plots are shown for different values of the repulsive on-site interaction strength $U$: $U > 0$.

In the experiments $U$ can be tuned by using the technique of Feshbach resonances.
Numerical results (II)

\( N = 100 \quad \lambda = 0.1 \)

\[ \begin{align*}
U/J &= 1 \\
U/J &= 0.2 \\
U/J &= 0
\end{align*} \]

- 1D DGPE
- 1D DNPSE

atoms per site vs. lattice site
Numerical results (III)

$N = 100 \quad \lambda = 0.1$

\begin{align*}
\text{1D DGPE} & \quad \text{1D DNPSE} \\
\text{U/J = 2} & \\
\text{transverse width} & \\
\text{lattice site} & \end{align*}

\begin{align*}
\text{atoms per site} & \\
\text{lattice site} & \end{align*}
Now we show the results obtained again with $N = 100$ atoms in a quasi-1D optical lattice but with an **attractive** on-site interaction strength $U$: $U < 0$.

In the attractive case the ground-state is self-localized and it exists also in the absence ($\lambda = 0$) of the axial harmonic potential: discrete bright soliton.
Numerical results (V)

N = 100  \quad \lambda = 0.1

U/J = -0.02

-20 -15 -10 -5 0 5 10 15 20

lattice site

atoms per site

0 10 20 30 40 50

-20 -15 -10 -5 0 5 10 15 20

lattice site

transverse width

0 0.5 1 1.5 2

-20 -15 -10 -5 0 5 10 15 20

1D DGPE
1D DNPSE

U/J = -0.02

N = 100  \quad \lambda = 0.1

U/J = -0.02

-20 -15 -10 -5 0 5 10 15 20

lattice site

atoms per site

0 10 20 30 40 50

-20 -15 -10 -5 0 5 10 15 20

lattice site

transverse width

0 0.5 1 1.5 2

-20 -15 -10 -5 0 5 10 15 20

1D DGPE
1D DNPSE

U/J = -0.02
Numerical results (VI)

$N = 100 \quad \lambda = 0$ (no axial trap)

$U/J = -0.02$

1D DGPE
1D DNPSE

$U/J = -0.02$
By increasing the attractive on-site interaction $U$ ($U < 0$) the 1D DGPE shows that eventually all the atoms accumulate into the same site.

Actually, the 1D DNPSE shows something different: before all the atoms populate the same site there is the collapse of the condensate: 1D DNPSE does not admit anymore a finite ground-state solution.

Numerically we find that the collapse occurs when $U < 0$ and

$$\frac{|U|N}{J} \gtrsim 2.1$$

which is consistent with analytical result\(^4\) $|U|N/J > 8/3$ of the continuum limit.

Collapse of the discrete bright soliton (II)

\[ N = 100 \quad \lambda = 0 \text{ (no axial trap)} \]
From the 3D GPE of bosons in a quasi-1D optical lattice we have derived an effective 1D DNPSE. The DNPSE reduces to the 1D DGPE in the weak-coupling limit. The DNPSE gives quite different results with respect to the 1D DGPE in the (repulsive) strong-coupling limit. In the case of attractive on-site interaction there is a self-localized solution: the discrete bright soliton. The DNPSE predicts the collapse of the discrete bright soliton above a critical (attractive) on-site interaction. Our results are reliable in the superfluid regime $|U|N/J \ll N^2$ where the 3D GPE is meaningful.
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